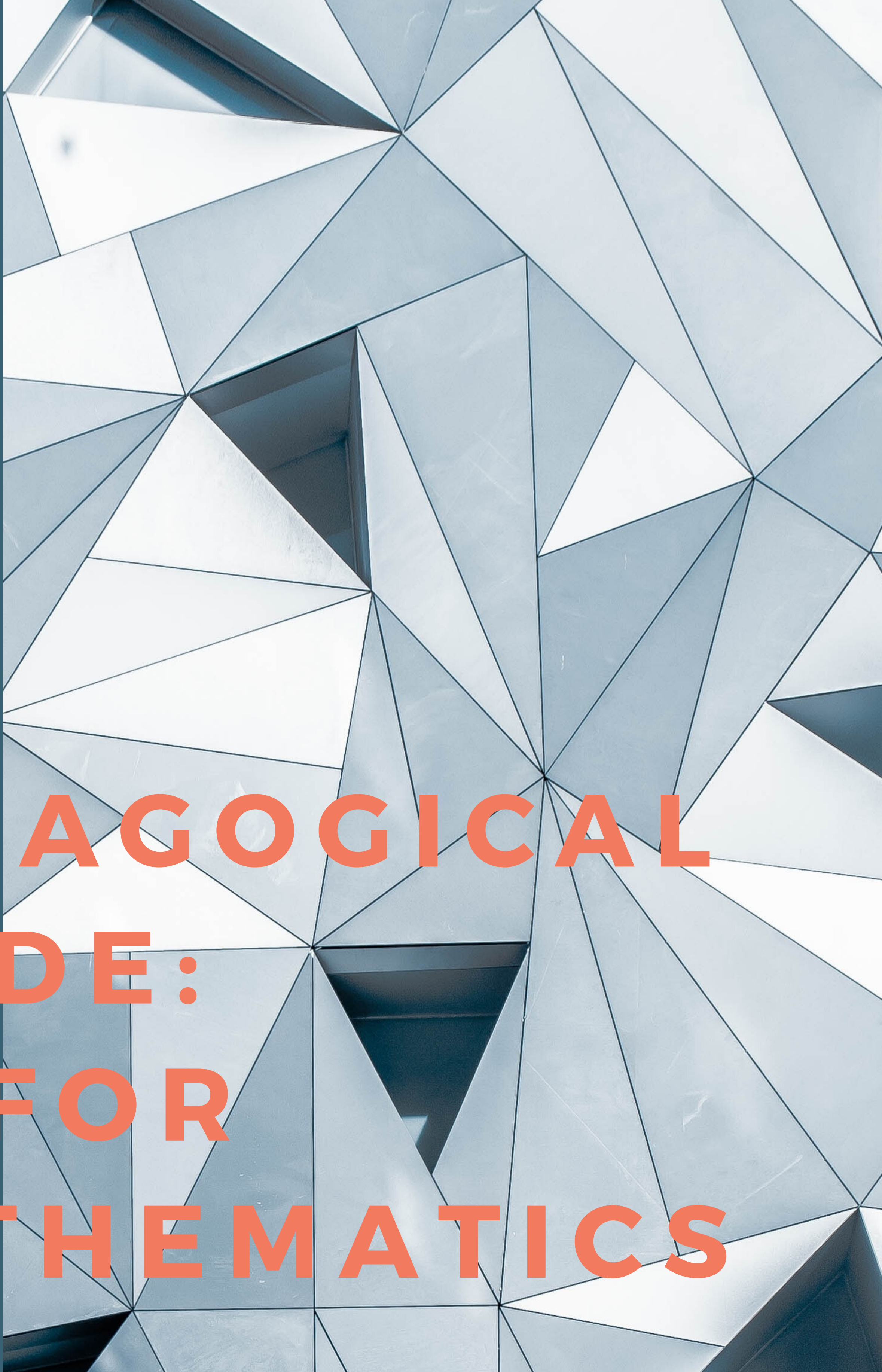
PEDAGOGICAL GUIDE: VR FOR MATHEMATICS



Math Reality

The only source of knowledge is experience

ALBERT EINSTEIN

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THE NON-FORMAL APPROACH OF TEACHING MATHEMATICS

INTRODUCTION; WHAT IS THE NON-FORMAL APPROACH OF TEACHING MATHEMATICS

Non-formal education: comprising an educational activity organised outside the formal system and designed to serve identifiable clientele and educational objectives
Coombs, Prosser and Ahmed, 1973

Non-formal education has several of the inherent characteristics of formal education, as they both share a commitment to learning and knowledge acquisition, thus being developed according to methodologically designed curriculum and scientifically sound resources. However, there are many points of non-convergence; the most obvious being that formal education takes place in a school building, while non-formal education can occur in any place that belongs to a community. Accordingly, non-formal education may use clubs, camps, group meetings, sporting or arts activities, or youth-led events for carrying out educational work, thus occurring in a variety of social and communal environments and in diverse forms.

"Democratic Education" through experiential learning

Non-formal education could be developmentally beneficial in various ways. As Van Horn, Flanagan and Thomson claim (1998) non-formal education promotes the experiential learning, the privilege of personal choice and it scatters different types of interpersonal relationships. Through the structured assignment of work, such as creative tasks and activities, young people are encouraged to take decisions related to the mode of working they prefer to be led to the successful absorption of knowledge, thus feeling flexible to extensively investigate their abilities and some of their emerging interests.

The basis of non-formal education underlines the connection of individuality with the community, in a way that all the activities may respond to the needs and demands of the individuals, but within the challenges that the social community itself calls for (Carver, 1998). In this sense the individual and the community have a mutual relationship of giving and taking.

Enfield (2001) also argues that the parameter of experiential learning nurtures the development of skills and knowledge, as it is designed in a fully engaging way that fosters both inter-personal skills and interactivity. He also claims that experiential learning boosts self-confidence within creativity and cultivates personal relationships in non-formal settings, not only between young people, but also among youth and adults. Additionally, research findings indicate that, apart from the cognitive
level, non-formal methods of education 'foster positive youth development, regardless of the method, setting, or backgrounds of the youth involved (Russel, 2001).'

Under such experiential process, young people gain the opportunity to develop a series of soft skills -implying the possibility to explore personal skills, competences and values which are not always easily detectable within the educational framework of formal systems- such as: organizational management, teamwork, conflict management, 'planning, co-ordination and organizing', self-confidence and self-esteem, practicality, responsibility, leadership, sharpening of the ability to solve problems in a practical way, discipline, intercultural awareness and many other soft skillscorrelated with global education.

To conclude with, non-formal education seems to have a beneficial influence in four basic pillars that are interwoven with youth's life:

In personal development: it helps young people to emphasize their self-confidence and self-esteem, to realize their strengths and weaknesses, thus being encouraged to act outside the district limits of their comfort zone, as well as to discover the range of their abilities, gifts and talents;

In the development of active citizenship: it cultivates social skills and competences related to citizenship, as well as to the expression and understanding of different opinions in our increasingly diverse societies. It acquaints young people with important social and political concepts and structures, as well as with the duty of active and democratic participation;

In the abetment for employability: It is perhaps the best way to acquire the horizontal skills that the labour market requests: critical and creative thinking, initiative, problem solving, risk assessment, decision making, constructive management of emotions and resilience and;

In the formation of more human societies, as it brings people closer, constituting a powerful force in shaping human behaviour. It builds the capacity for empathy, which helps in understanding both the mentality and feelings of other people. And this ability increases the social sensitivity of individuals, thus fighting against stereotypical behaviours and other inter-related phenomena such as intimidation, prejudice, racism.

Generally, non-formal methods of learning maintain a vibrant tradition in Europe, as since 1990's they have been considered as the Council' s of Europe main methodology -along with its inherent philosophy- in terms of the youth programmes that are being organized with the exclusive usage of European funds.

Learning Mathematics in non-formal educational settings

At present, studies such as of Carraher and Schliemann (2002) reaffirm the idea that non-formal mathematics can provide a basis on which learners can really rely to in building more sophisticated mathematical knowledge. These two authors consider that classroom non formal activities should allow the learner to experiment with a plurality of mathematical situations, tools and concepts that make explicit the linksbetween the mathematics of everyday life and those developed in school.

With non-formal mathematics, the learner is at the heart of learning: he discovers, manipulates and models. They can be based on individual and group learning as part of an overall collective approach, they are participatory and learner-based, they are action and experience-based.

Non-formal mathematics can therefore demystify math in order to give it a taste from an early age and, therefore, encourage STEM (Science, Technology, Engineering and Mathematics) to contribute to the economic development of our countries.

TOOLS FOR LEARNING IN NON-FORMAL EDUCATION

What's a Tool for Learning?

As we shall see later on, it is rather difficult for one to figure out what the notion 'non-formal methods' really implies, by merely focusing on its clear definition; in fact, it would be much easier to try to comprehend those 'non-formal methods' through their features, whilst having in mind that they could concisely be classified in four sub-categories:

- The communication-based methods, based on interaction, dialogue and mediation;
- The activity-based methods, based on experience, practice and experimentation;
- The socially focused methods, based on partnership, teamwork and networking and;
- The self-directed methods, based on creativity, discovery and responsibility.

[Source: Council of Europe Symposium on Non-Formal Education: Report (2001)]

Accordingly, if the educator/ teacher/ trainer intends to use one, or even a combination of non-formal methods which lie on the previous categories in order to facilitate the learning process of a mathematical concept, first and foremost, he/she should design a complete educational tool which will fundamentally be comprised of non-formal methods.

For that reason, we consider that it is important for now, to explain, what is a tool for learning, how do we recognise it and which criteria a tool should satisfy. Hence, after providing those theoretical information, along with three analytical examples of mathematical tools (one for each age-group separately) consisting of non-formal methods of education, the educator/ teacher/ trainer dealing with the teaching of mathematical notions, will be capable to design and construct his own mathematical tools for learning.

To start with, an educational tool is, above all, an 'instrument' that we can associate with an approach and which is being developed in order to facilitate a learning process. In other words, the tool constitutes a 'medium' which is built on the basis of a methodological and educational approach, with the prime aim of helping 'a public' to understand or to learn, thus absorbing new knowledge.

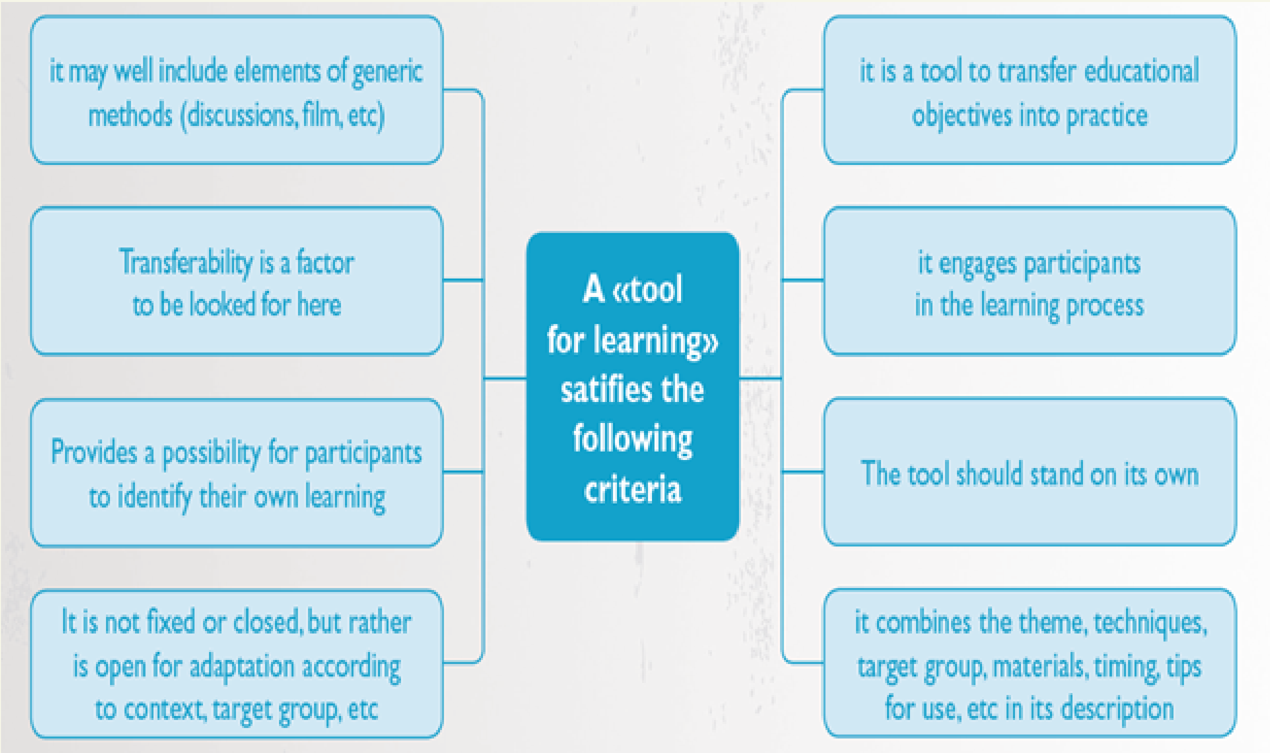
Therefore, an educational tool carries the capacity to transform clearly defined educational objectives into practice, whilst engaging the target group in the learning process. One of its more significant characteristics is that a tool must be capable of 'staying on its own'. That means that the tool should develop a maybe short, but fully complete educational process, which does not need any other additional external details in order to make sense.

The fact that a tool should 'stay on its own', does not necessarily imply that it can never be altered; contrariwise, an educational tool should be mutable and 'open' enough, in a way that allows to be used in various contexts, always giving the possibility of being adapted, combined and further developed, depending on the real conditions and on the environment in which it's being applied. This inherent characteristic of a tool- which in parallel constitutes one of its fundamental aims- is called 'transferability'.

Let's not forget that an educational tool combines non-formal methods which can be communication-based, activity-based, socially focused and/or self-directed methods. So, a tool for learning can be a simulation exercise such as a role-play, a workshop that fosters creativity and raises imaginary or fictional features, an activity that takes place outside, or an activity which invokes experiential processes and facts deriving from every-day life. An educational tool could be a game, an interactive e-video, a story, a discussion, a manufacture, a film, a photo or picture accompanied with a text, or even a combination of some (or all) of the previously mentioned things, always given in a logical order and in a way, that in fact facilitates the learning experience.

Principles for the conception of a Tool for Learning in Non-Formal education

As derived from the previous paragraph, a tool for learning in non-formal education is distinguished by particular features, which are subjected to certain criteria. Those 'criteria' or 'principles' that a tool should obey to, are presented in the following diagram:



Source: Council of Europe Symposium on Non-Formal Education: Report (2001)

In accordance with the diagram, a good tool for learning in Non-formal education must combine all of the eight criteria at the same time. Consequently, when the educator/trainer/ teacher intends to create his own educational tools, he could use the previous diagram as a checklist in order to be certain that his tool is a successful one, inasmuch as its construction is based on valid and objective principles.

In addition, some other useful points that must be taken into consideration during the constructive process of an educational tool are summarized below:

- A tool should give a sense;
- A tool should be easy to use by everybody (or by the target group has been made for), without excluding anyone from the learning process;
- A tool should be written in a precise language and it should convey clear messages in a vivid way;
- A tool should facilitate the processes of addition of information, modification and adaptation;
- A tool should be dynamic and appealing, while focusing on interactive processes;
- A tool should be balanced and should promote the notion of balance, serving at the same time both individuality and collectiveness;
- A tool should provoke the learners to push their limits, by being forced out of their comfort zone, without though being frightening or seemed unreachable;
- A tool should constitute an invitation to travel in any possible space through the paths of imagination and creativity.
- A tool should serve the idea of ‘learning by doing’.

Teaching Aid; Media and Techniques

During the procedure of construction of an educational tool, the educator/ teacher/ trainer should keep in mind that a concrete methodology has to be followed, comprised of clear steps, which could probably obey to the following draft:

Title/Age Range or Target Group/Duration/Mathematical Concepts that the tool deals with/ General Aim/Objective/Instructions for use (Step-by-Step process)/Materials and Resources/Media/Techniques and Methods/Tips for the educator/Desirable Outcomes and Competences/Debriefing/ Questions for Evaluation.

Accordingly, after being sure about the *Mathematical Concepts* that the tool would deal with and taking final decisions on the *General Aim* and *Objectives* of the tool, the educator might be ready to write down *Instructions for Use*, by giving in an analytical way the *Step-by-Step process*. Even before finalizing the *Step-by-Step process*, the educator/teacher/trainer can define the *Teaching Aid* he/she will intend to invoke. And by saying Teaching Aid, we imply the teaching support which is primarily composed of all the *Media* and *Techniques* that will be included in the process of a tool's design and construction. At this stage, it would be good to note that, frequently, the *Media* is being given in a way that is mixed-up with one or more Techniques; in other words, it's not always achievable to distinct the Media from the corresponding *Technique(s)*.

Taking all the above into consideration, a list with the most frequently used Media and Techniques is being provided in the following Table:

TEACHING SUPPORT OR AID:	MEDIA	TECHNIQUE	MIXED MEDIA AND TECHNIQUES
	White board or Paperboard	Use of White Board or Paperboard in order to present, observe and/ or summarize a new knowledge etc.	
	Computer	Use of Computer in order to present data, to complete a task, to conduct research on a topic, to use a specific program etc.	
	E-Video	The projection of an E-Video in order to give information on a topic or a concept. It could be a documentary, a film etc.	
	Video Projector	Use of Video Projector in order to facilitate the process of a video presentation.	
	Book	Use of a Book in order to say a story, to introduce a new concept, to give new knowledge, to provide historical or other features, to cultivate a positive stance towards a topic or a concept, to cultivate imagination, to broaden the field of knowledge, etc.	
	Photos or Pictures with or without Text	Use of photos in order to convey a vivid message, to represent a concept or a task, to give explanatory information, to make the instructions more precise, etc.	
	Graphs, Tables, Diagrams, Maps	Use of Graphs, Tables, Diagrams and Maps in order to depict information, to provide features in a more readable way, to map an order of information.	
	Graphics	Use of Graphics in order to make the presentation of other Media more attractive, to add meaning in a text, etc.	
			Games: To learn easily, to absorb a new concept, to show the applicability of a concept, to cultivate a positive stance towards a concept, etc.
			Stories/Storytelling: To introduce a new concept, to give new knowledge, to provide historical or other features, to cultivate a positive stance towards a topic or a concept, to

			cultivate imagination, to broaden the field of knowledge, etc. Case Study: To provide a new knowledge through giving a concrete example, to add new possible methods of making a research, to learn how a theory is being applied, etc. Simulation Exercise (such as Role-Play): to introduce and absorb a knowledge within a creative way, to put one in a position of another person, to cultivate empathy and compassion, etc. Group Discussions and Presentations: To urge the exchange of opinions and knowledge, to learn how to raise good argumentation, to work on communication skills, to build-up one's own methodology in discovering and presenting something, to learn how to make research related to a specific topic, to learn how to put the things down in well-organized way

Table: Teaching Support or Aid; Media and Techniques

AN EXAMPLE OF NON-FORMAL TOOLS RELATED TO MATHEMATICS WHICH COULD BE INCORPORATED IN THE OFFICIAL MATH CURRICULA

AN EXAMPLE FROM EU-FUNDED PROJECT 'INFORMATH' PROJECT

In this section, we provide the reader with a good example of a non-formal tool, namely "The Warrior Student: An adventure in Calculus" which has been retrieved from the Erasmus+ Project 'Informath'. This specific tool constitutes a game-book that helps students (16-18) to test their abilities to analyze functions.

"The Warrior Student: An Adventure in Calculus"

Age Range– Target Group
The age range of students is 16-18-year-old secondary school students. Students must have been taught calculus and the study of functions.

Duration
The duration of the game is up to 30 minutes, depending on the students' ability in the Calculus.

Mathematical Concepts that the tool deals with
The game deals with the mathematical concepts of functions, graphs, monotonicity, differentiability, sign of function, extrema and concavity.

General Aim-Objective
The "Warrior Student" comic is a game-book that helps students to test their abilities to analyze functions. During the game, students will have to analyze a function. The analysis is divided into several steps. Many common and basic mistakes on the study of function are integrated in the story. So, maybe those mistakes will be avoided by the students, since they could visualize them with a graphic and easy to memorize math-adventure. With this comic, students analyze functions in a fun way.

Soft skills related to game
Problem solving, self-confidence, decision making.

Materials and Resources
The "Warrior Student" comic game is a small-sized book of 44 pages in the form of a booklet. Every student who participates in the adventure should have at his disposal a pen or pencil and a paper, in order to answer the quiz questions and dilemmas.

Step-by-step process
In general, for the game-book:
The scenario of the mathematical adventure book is unfolding into two parallel worlds: a world where the reader is a student who solves an exercise and a world where the reader is a warrior in an adventure. Every student who participates in the adventure should read the comic and solve the puzzles / dilemmas / problems correctly. Depending on which answer the student gives at each stage, the comic directs the reader to a different page that the fighter will win or lose. Thus, if the reader, as a student, follows the steps of the solution properly, then the fighter responds positively to the trials. If the reader, however, as a student makes mistakes then fighter faces problems in the adventure.

Any wrong choice removes energy points. Student may lose after many mistakes that eliminate the points. Student's initial life points are five. Throughout his/hers adventure an assistant and coach is an owl (figure 1).

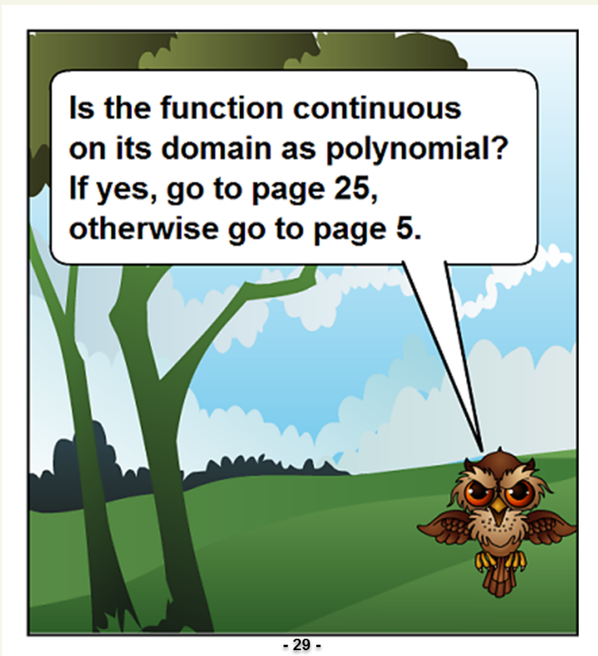


Figure 1. The owl sets directions for the student.

The story of the student reader is the task to study and plot a function (figure 2).

$$f(x)=\frac{x^2+x-1}{1-x}$$

Figure 2. The function that the student will have to study.

The story of the warrior reader is to face the imaginary team of Contrapositive. This group consists of Ninja Osho, the Caveman, the Gorilla Iverne and the Big snake (figure 3). All four will try to stop him. Warrior must be prepared all time.

Figure 3. The team of Contrapositive, the opposing team.

If the warrior succeeds in completing the adventure without zeroing the energy points, then the Warrior Student wins.

The adventure of the comic unfolds from one page to another, but not in sequentially numbered route (the reader does not easily see the continuation of the story on the next page).

The game is mainly individual but can also be played by a team of two students.

A hypothetical case of play

Assume the following case: the reader is on page 18 and student must find the domain of the function. The owl defines A=R-{-1} as the domain. If the reader has found the same domain, then the reader will continue on page 30, otherwise the reader will continue on page 36 (figure 4).

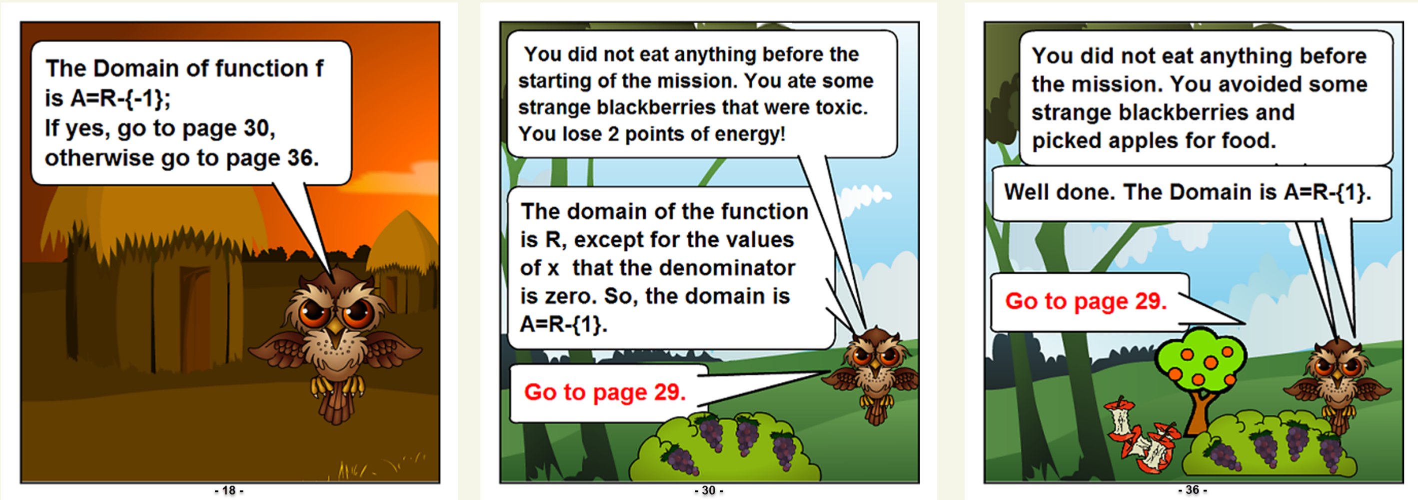


Figure. 4. A dilemma, a bad and a good mission progression.

So, if the student goes to page 30 (wrong choice), warrior will face problems, and if the student goes to page 36 (the right choice), warrior will continue unscathed.

Teaching Methods and Techniques

The game is based on a category of books that are called adventure gamebooks. In the version of this book, teacher added a parallel world, the world of student who with his answers affects the world of a warrior. Also, instead of plain text, the action is represented by comic images (using the internet platform toondoo).

The teaching methods and technical methods that are applied:

- Reading.
- Individual solution finding.
- Play game.
- Individual evolution of the story depending on student's choices.
- Use of graphics and comics.
- Reward for correct answers and explanations when there is a wrong answer.
- Focus on common and basic mistakes with an attempt to visualize them with a graphic and memorable vision.

Tips for the Educator

In the future, it would be good to support the story of comic with more text, narrative and a better story, which can enhance students' imagination. As a part of an education student project, educator may ask students to produce a comic like "Warrior Student", using comic software (the toondoo or other).

Questions for Evaluation

In order to improve this mathematical game, it should be evaluated. Possible questions that can be asked to students are:

Do you like the story of
the warrior student?

Was it difficult to browse the content of comic?

Do you think it helped you to see math from a different perspective?

Finally, do you prefer a typical exercise or an exercise through this type of comic?

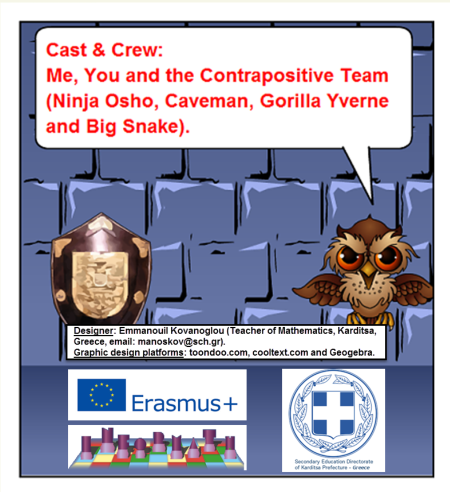
Resources

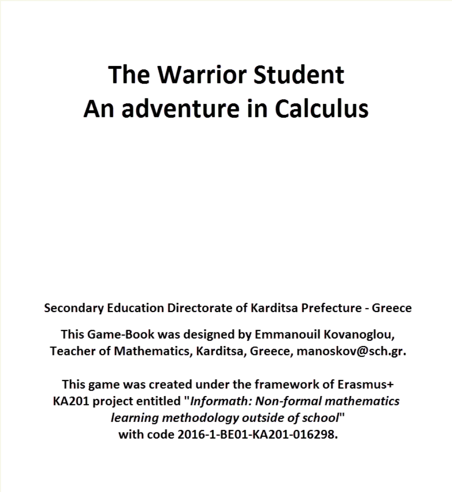
All frames of the comic were made via www.toondoo.com.

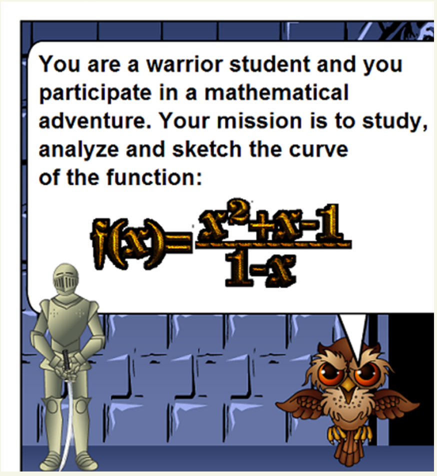
The graphs were made with Geogebra.

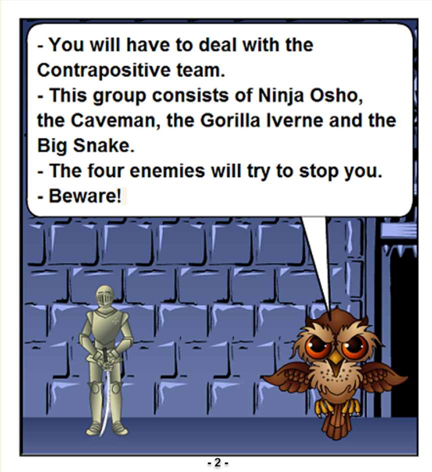
Title font was made via www.cooltext.com.

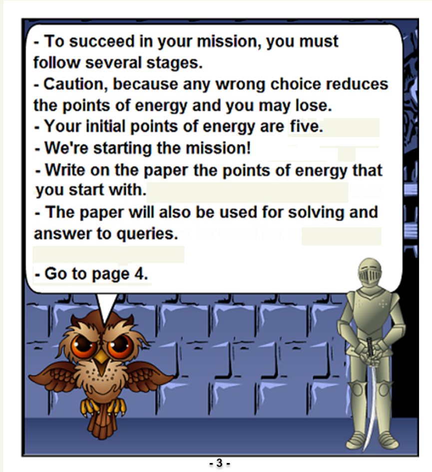
Material (pages of the booklet)

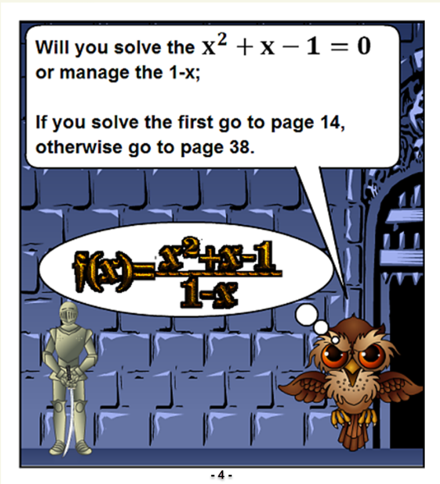


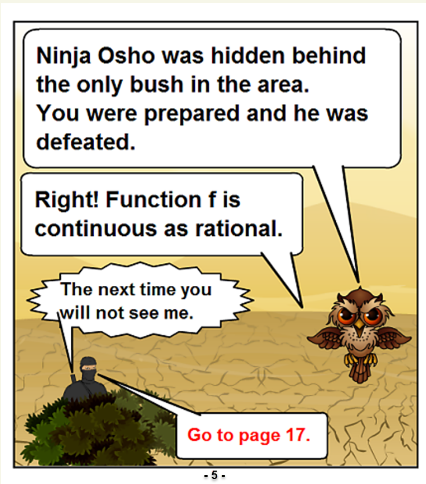


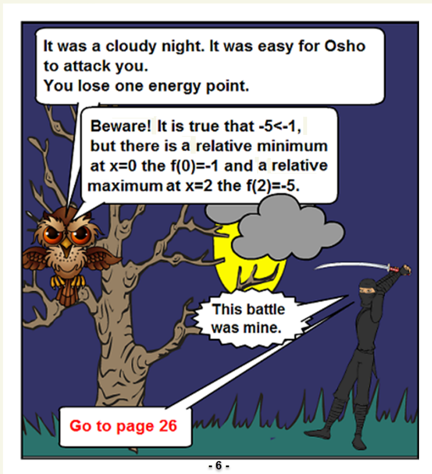

You are a warrior student and you participate in a mathematical adventure. Your mission is to study, analyze and sketch the curve of the function:
$f(x)=\frac{x^2+x-1}{1-x}$

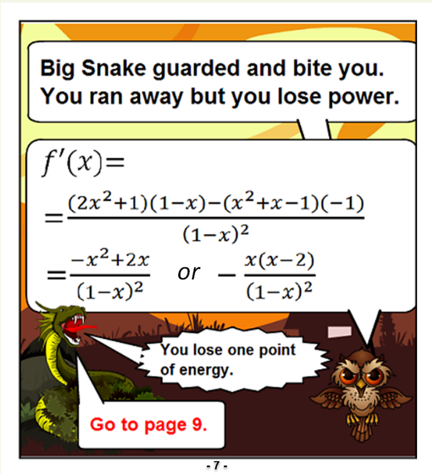

- You will have to deal with the Contrapositive team.
- This group consists of Ninja Osho, the Caveman, the Gorilla Iverne and the Big Snake.
- The four enemies will try to stop you.
- Beware!
- 2 -

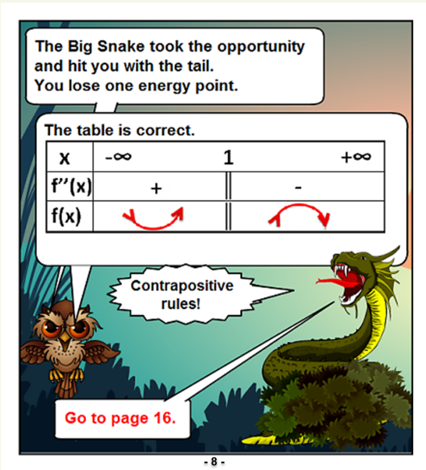

- To succeed in your mission, you must follow several stages.
- Caution, because any wrong choice reduces the points of energy and you may lose.
- Your initial points of energy are five.
- We're starting the mission!
- Write on the paper the points of energy that you start with.
- The paper will also be used for solving and answer to queries.
- Go to page 4.
- 3 -

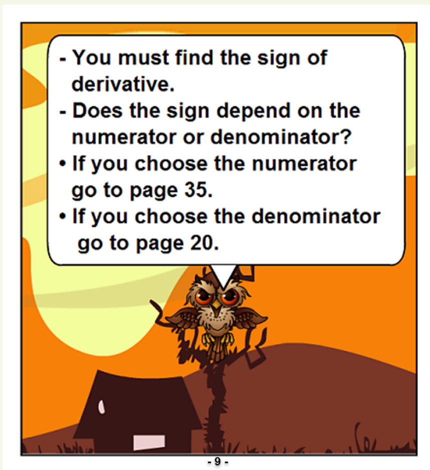

Will you solve the $x^2+x-1=0$ or manage the 1-x;
If you solve the first go to page 14, otherwise go to page 38.
$f(x)=\frac{x^2+x-1}{1-x}$
- 4 -

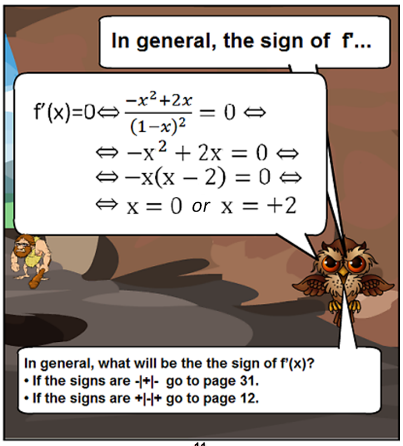

Ninja Osho was hidden behind the only bush in the area. You were prepared and he was defeated.
Right! Function f is continuous as rational.
The next time you will not see me.
Go to page 17.
- 5 -


It was a cloudy night. It was easy for Osho to attack you. You lose one energy point.
Beware! It is true that -5<-1, but there is a relative minimum at x=0 the f(0)=-1 and a relative maximum at x=2 the f(2)=-5.
This battle was mine.
Go to page 26
- 6 -


Big Snake guarded and bite you. You ran away but you lose power.
$f'(x)=$
$=\frac{(2x^2+1)(1-x)-(x^2+x-1)(-1)}{(1-x)^2}$
$=\frac{-x^2+2x}{(1-x)^2}$ or $-\frac{x(x-2)}{(1-x)^2}$
You lose one point of energy.
Go to page 9.
- 7 -


The Big Snake took the opportunity and hit you with the tail. You lose one energy point.
The table is correct.
x | -∞ | 1 | +∞
f''(x) | + | ‖ | -
f(x)
Contrapositive rules!
Go to page 16.
- 8 -


- You must find the sign of derivative.
- Does the sign depend on the numerator or denominator?
• If you choose the numerator go to page 35.
• If you choose the denominator go to page 20.
- 9 -



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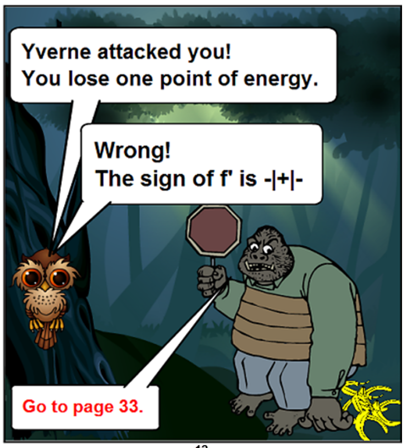

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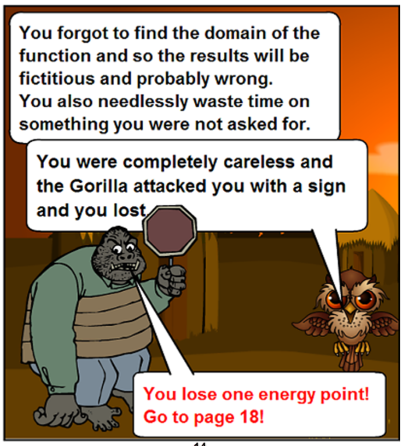

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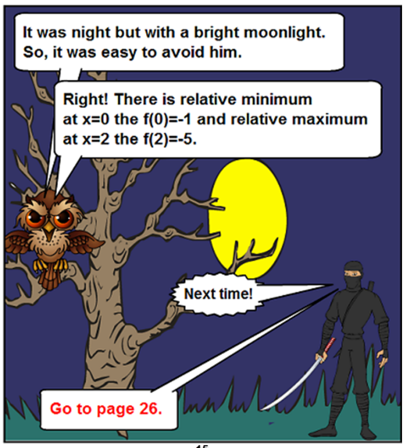

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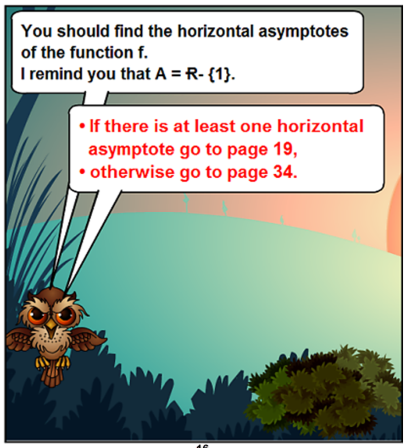

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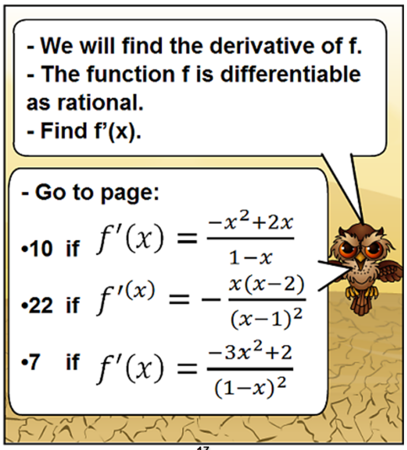

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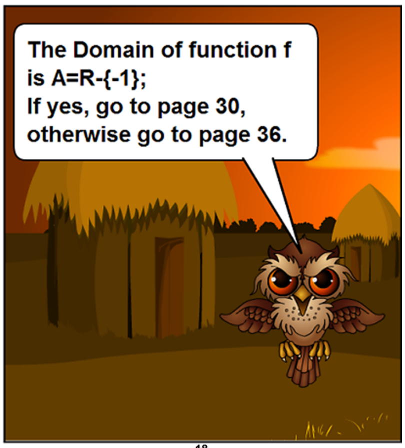

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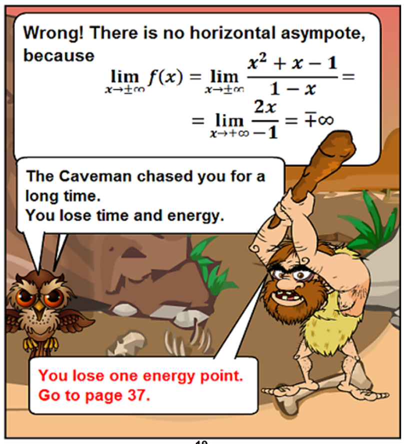

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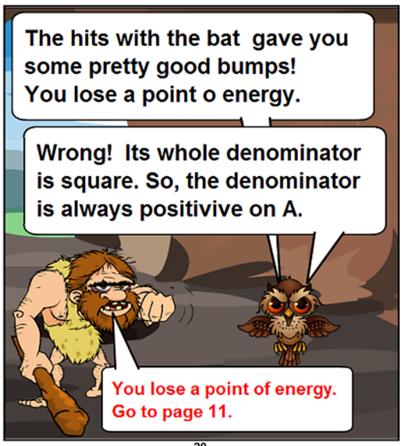

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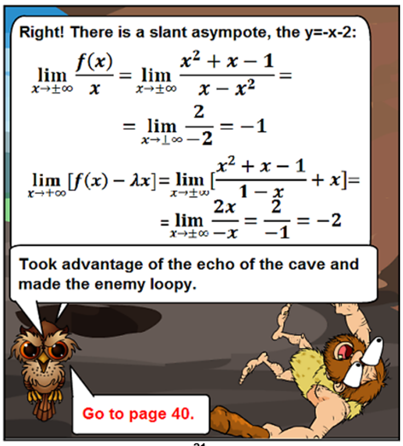

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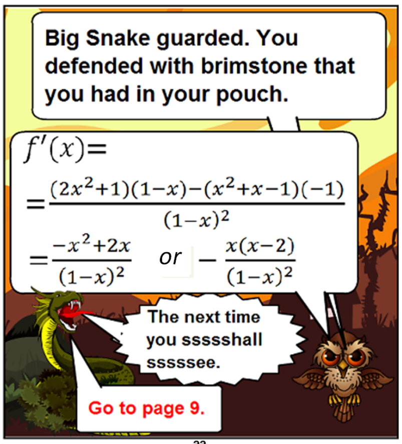

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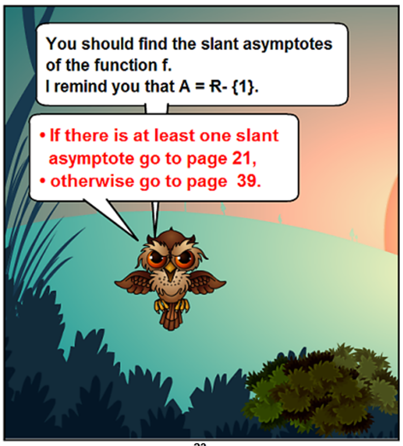

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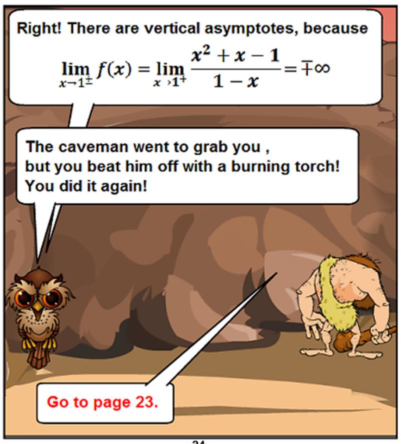

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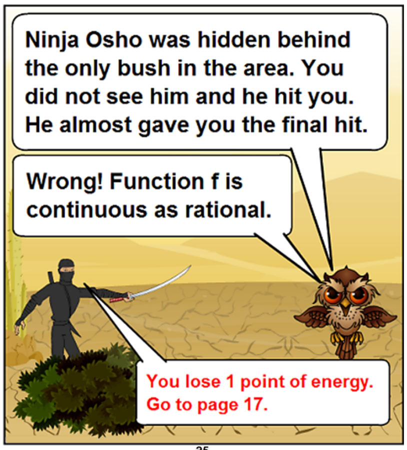

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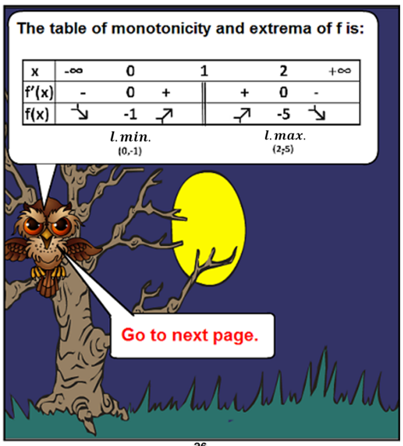

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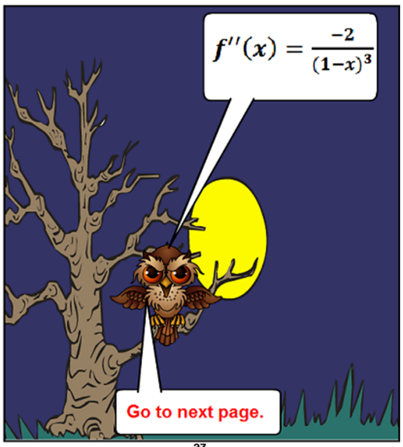

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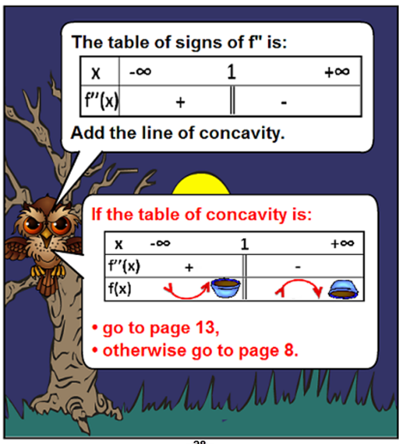

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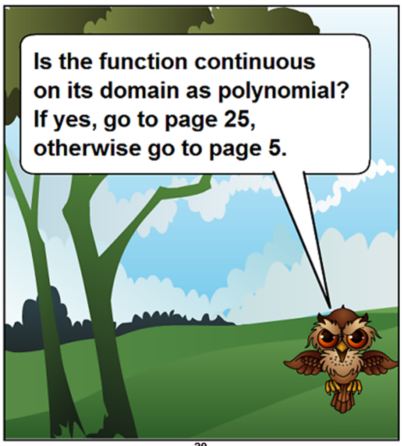

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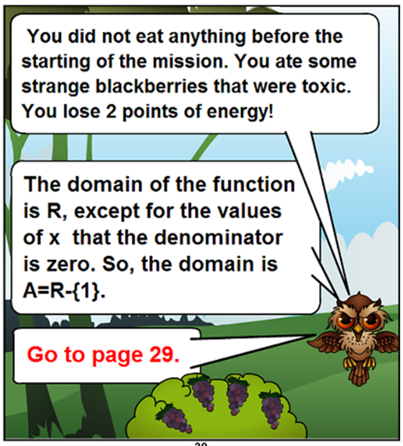

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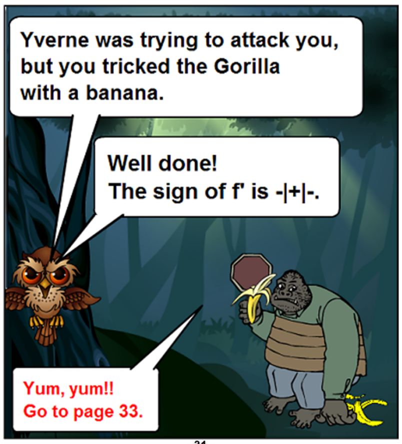

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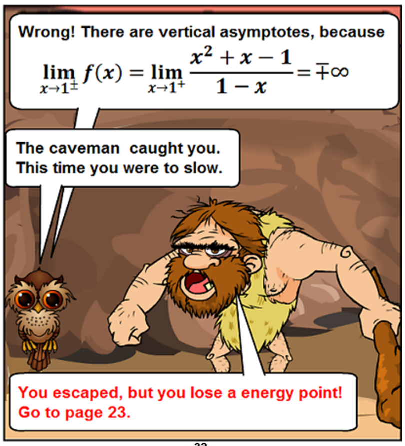

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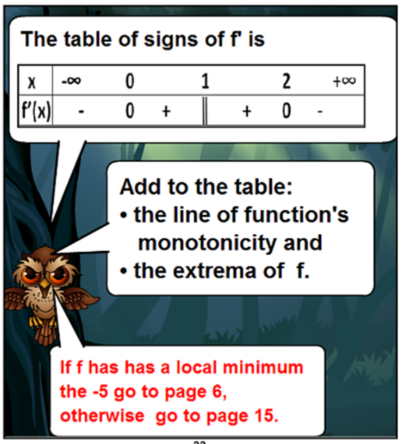

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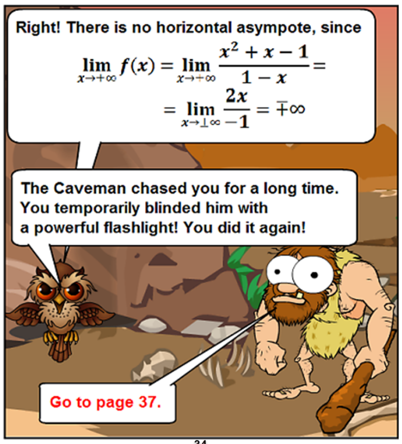

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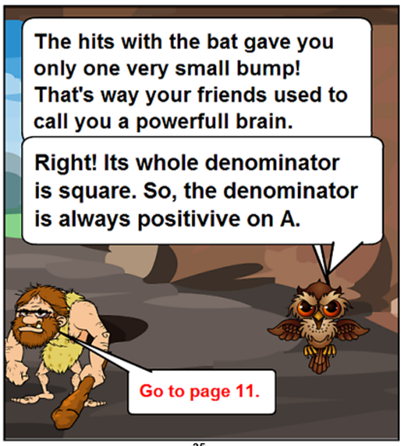

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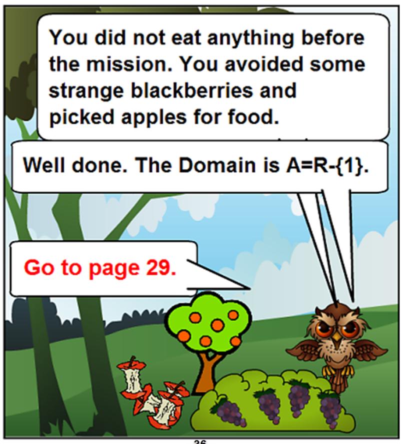

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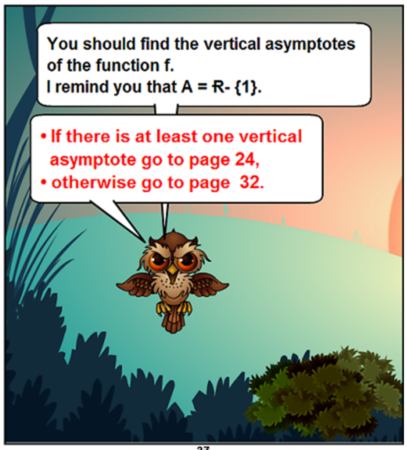

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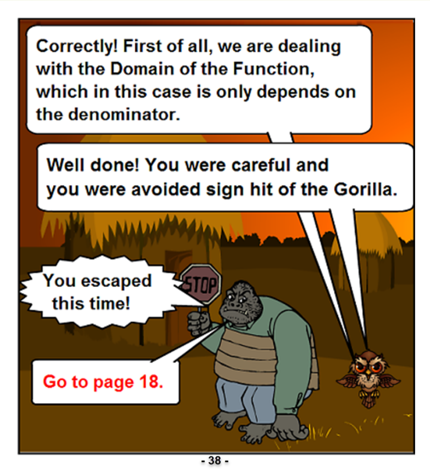
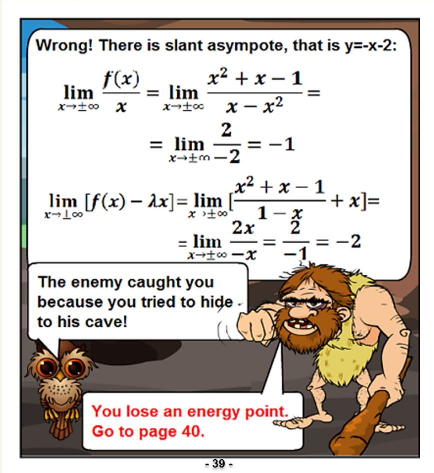
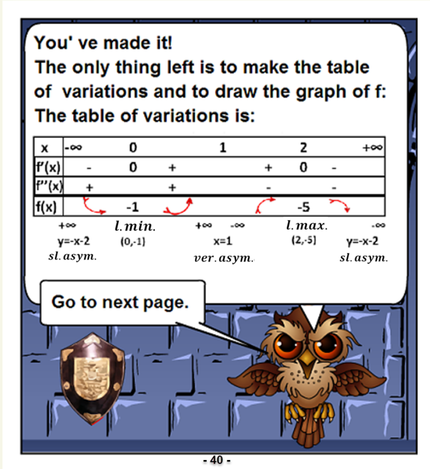
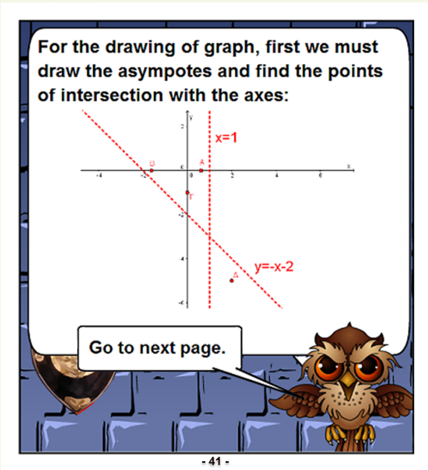
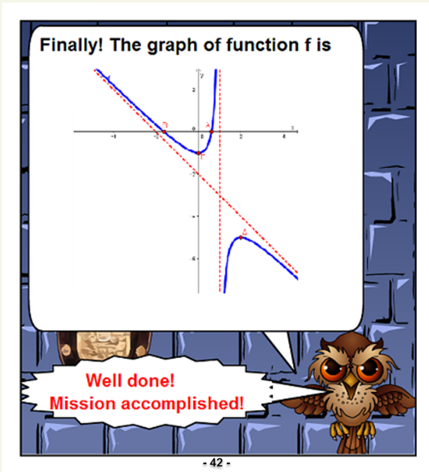










AN EXAMPLE FROM THE EU-FUNDED PROJECT 'MATHSPACES'

A great percentage of kids start mathematics with the idea that mathematics is difficult, harsh. To counter that, there is a movement emerging in the last 10 -15 years all around Europe with the creation of museums, centres or houses of mathematics to promote a non-formal approach of mathematics who has a proven effect on the mathematic skill of the youngsters, and above all on their engagement with mathematics.

However, the spaces dedicated to this approach of maths are still only a few. This shortage is due to the lack of knowledge about the approach and the difficulty to find the appropriate resources and content.

So the partners of the MathSpaces' project decided to create a project that aims at increasing the awareness and practice of the increase of the spaces dedicated to non-formal approach of mathematics in the European Union. In order to do that, partners wanted to create:

1. A booklet on the efficiency of the non-formal approach of mathematics and a practical Guidebook on the creation of spaces dedicated to the non-formal approach of mathematics.
2. A database of tools, games, activities and stations available in Opensource
3. The first 2 math exhibitions, scientifically and pedagogically accurate, in Opensource with blueprints and tutorials.

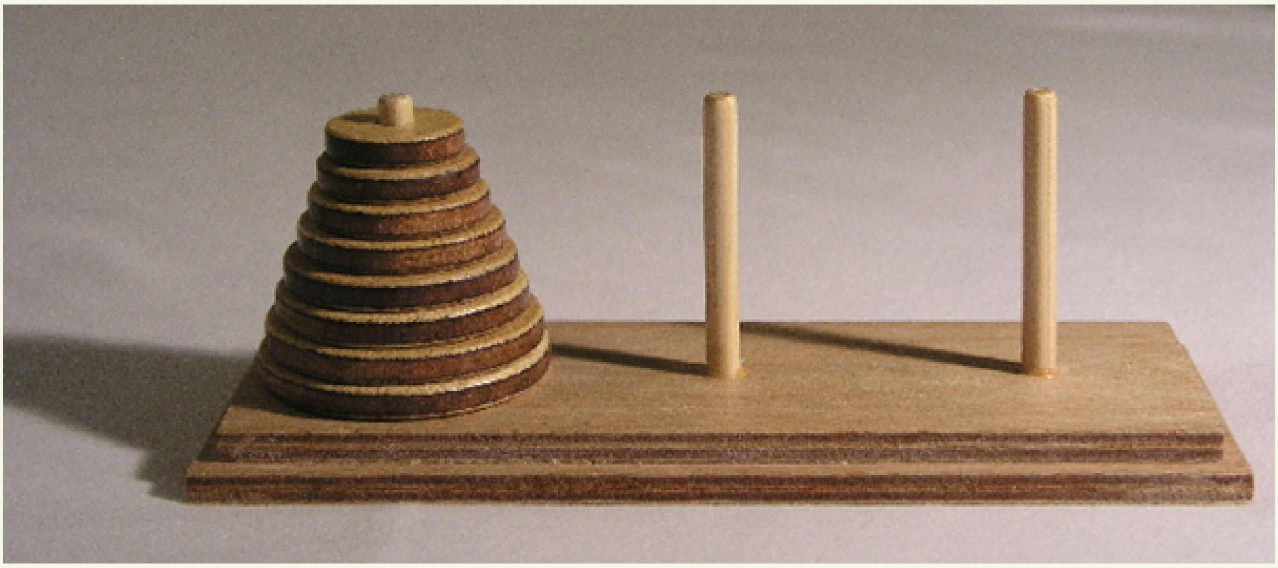
The booklet is available in English on the website of the project : http://mathspaces.eu/

The leader of the MathReality project, Fermat Science, participated in the development of this project MathSpaces. As far as the other tools, they are postponed at the moment due to some barriers but some of the non-formal tools already developed can be analysed here as examples.

Brainteaser and maths

The Tower of Hanoi

This is a brainteaser invented by French mathematician Edouard Lucas (1842-1891), devised in 1883. Several discs of different sizes are stacked on a rod in decreasing order so the largest is at the bottom and the smallest is at the top.



Source: https://en.wikipedia.org/wiki/Tower_of_Hanoi#/media/File:Tower_of_Hanoi.jpeg

The aim of the game is to move this mower to one of the two other rods. To do this, you can only move one piece at a time, without ever putting a bigger disc on top of a smaller one. All three rods must be used.

How many moves?

The number of moves required to complete a Tower of Hanoi can be calculated using the formula 2n – 1. So, two discs need to be moved three times, three discs need to be moved seven times, etc.

Demonstration - maths

The demonstration is very simply by recurrence.
-The result is true for n = 1.
-Suppose the true result for n and show that it is true for n + 1:
So, we have 2n - 1 moves for n discs.
For 1 disc more, so for n + 1 discs, it will take 2 (2n - 1) +1 moves;
2 (2n- 1) +1 = 2n+1 - 2 + 1 moves, that is to say 2n+1 - 1 moves.

Finally, the result is true for 1 disc; when it is true for n discs, it is true for n + 1 disc. The recurrence is well demonstrated: for n discs, we have 2n - 1 moves.

More - maths
The problem of Hanoi towers is seen in algorithmic (programming), where it offers an example of the power and readability of programs defined recursively (another example being tree sorting).

The Hanoi Towers can be also represented by an abstract graph, each vertex of the graph representing a possible arrangement of the N discs on the three towers, an edge connecting two vertices if there is a movement of a disc allowing to pass from a layout, represented by one of the vertices, to the other.

Source: Thomas Ricaud, Fermat Science

The shortest way

In the MathSpaces' project, the intellectual output n°5 is a consistent and scientifically correct exhibition in Open source for the youngsters from 9 to 15 years old.

For this category, Math starts to be more specific, it is not all about games and basic logic anymore but it is not yet about "technical", pointy theories. The learning at that age is focused on the learning of the great theories and concept on which modern mathematics are based on.

The theme chosen by the partner is The shortest way. The main aim is for the children to be introduced to some 'minimum' problems relevant in everyday life.

In an open space, the shortest path connecting two points is a straight line. But what happens if we are on the surface of a sphere?

Or if there is a flowerbed in the middle with the sign «Keep off the grass»? Or if we must connect three or more points? This exhibition is about these and other similar situations.

Below, you can find the modules under construction for the MathSpaces Project:

Module 1 Geodesics

In differential geometry, a geodesic is a generalization of the notion of a "straight line" to "curved spaces". The term "geodesic" comes from geodesy, the science of measuring the size and shape of Earth; in the original sense, a geodesic was the shortest route between two points on the Earth's surface, namely, a segment of a great circle.

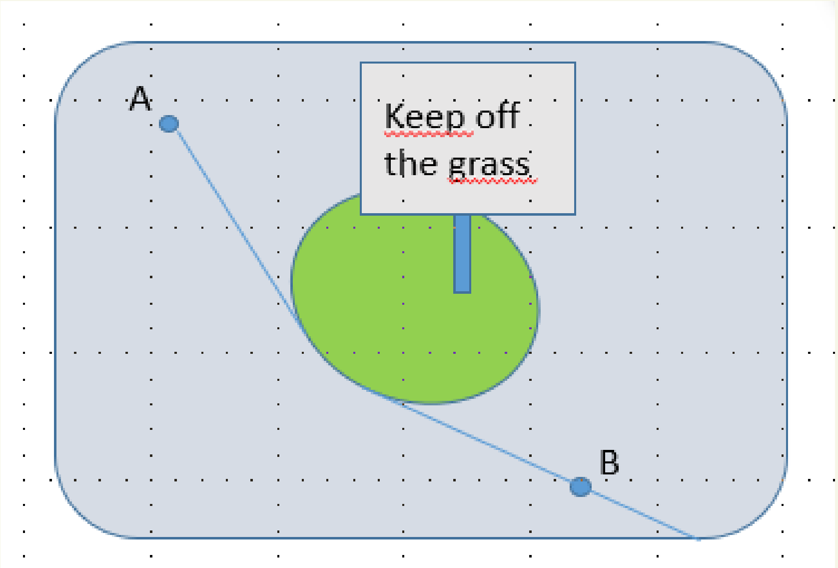
The term has been generalized to include measurements in much more general mathematical spaces; for example, in graph theory, one might consider a geodesic between two vertices/nodes of a graph.

What?

We can propose to the students an earth globe on which the visitor can experiment the shortest way between different cities. But also we can add a ceen with the program Mappa Mundi (https://mappaemundi.campus.ciencias.ulisboa.pt) which shows maps of the Earth in different projections and it can draw geodesics between any two points.

Module 2 A square with a flowerbed
The best path is tangent to the bed.

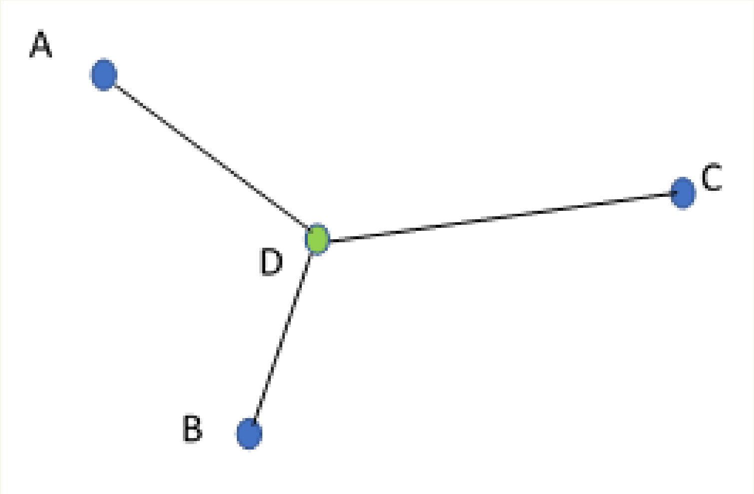
What?
A tape measure fixed in A measures the path between A and B.



Source: Alessandra Masala, Giardino di Archimede

Module 3 Connecting cities
How to connect three cities with a system of roads of minimum lenght? Given the three cities A, B and C, to find a point D such that the sum of the three distances AD+ BD+CD is minimal

What?
Two parallel plexiglas sheets are joined by small tubes. The object is dipped in a soap solution, and the result is as shown.



Source: Alessandra Masala, Giardino di Archimede

DO IT YOURSELF- CREATE YOUR OWN TOOLS FOR LEARNING NON-FORMAL MATHEMATICS;

As a careful reader could have observed, the information within the example of the previous part 1.3, is being given in a logical order, in order to facilitate both the practical materialization of the tool and the absorption of new knowledge. For that reason, it has been decided that all three examples should be re-structured and re-constructed under pre-determined subsections (titles), with clearer steps and sections as also mentioned above. Such structuring of information under the aforementioned titles could be particularly useful for the educator who intends to create his own tool for learning non-formal mathematics. Accordingly, this paragraph will give some more details on the process that one should follow in order to make his own tools.

Step 1: Select a target group

First, the creator of a tool for learning non-formal mathematics should decide on the target group that the tool will be addressed to. By saying 'target group', we imply the age range for which the tool will be suitable for, always in accordance with both the educational objectives and the curriculum as has been set by the school. In other words, we should always be sure that the material we intend to generate for a certain age range is consistent with the school curricula of the respective educational systems.

Step 2: Select a Mathematical Concept that the tool deals with

The second step for the creator of a new tool for learning non-formal mathematics is to decide on the mathematical concept that he/she would like to work on. A concept should be chosen that fits with the target group selected in the previous step and decide the extent to which it would be 'deepened' on the specific concept. Also, a matter of consideration should be if it is going to be connected with certain other interwoven concepts. Some issues to be addressed are if all the interrelated concepts have already been taught to the specific target group, and if they will need to be revised.

Step 3: Select the general Aim-Objective of your tool

As has already been mentioned, a tool must be able to transform educational objectives into practice. Accordingly, in order for the tool to be a successful one, the educational objectives set,should be realistic in terms of their materialization and clearly defined in terms of wording and formulation. Accordingly, both the creator and the recipient of the tool should have had clear in their mind, what will be the exact, desirable result after the completion ofthe exercise.

Step 4: Select a Time Duration

The creator should decide from the very initial phase whether his tool will last for a short or a long time, so as to 'fit' into it a reasonable 'amount' of tasks. Remember that a successful tool neither underestimates nor overestimates the parameter of 'time duration'. Besides, as one of the inherent characteristics of a tool is 'transferability', in some cases it might be better to indicate a time range, such as, for instance when the tool incorporates a game (see Nim_game – 15+).

Step 5: Select Media and Techniques

In the subchapter entitled as "Teaching Aid; Media and Techniques", a teacher can go through the Table 2.2 and make a selection of media, techniques, or even select an item from the third column, which indicates a combination of Media and Techniques. Your selection will inevitably depend on at least four parameters:

1) the educational material and resources that you currently have at your disposal. In case you lack such material, you can search for it in the net, thus coming up with some fresh ideas or even consult some relevant books and academic bibliography. In anyway, you could use the already existing material that is being given within already existing projects on STEM, and subsequently modify it according to your needs and demands, by merely following the instructions provided by the current section;

You should necessarily take into consideration:

2) the environment- social/ educational context under which the tool will be applied;

3) the educational infrastructures (computer rooms, open spaces, equipment) and potential access to new technologies and the net;

4) what it would better fit into both the mathematical concept you have selected, your target group, as well as what it serves your fundamental aim/objective. In any way, your tool could incorporate a combination of types of non-formal methods of learning.

Step 6: Write the instructions

During the authorship of the instructions, remember that a tool for learning should constitute a complete educational process, thus being able to 'stay on its own'. Accordingly, while recording the instructions of a tool, make sure that the instructions will follow a logical order, in order to avoid possible 'cognitive gaps' during the educational process; this implies that the creator should have built from the very basics of a concrete mathematical concept up to the more complex elements that the specific notion might contain. What is more, remember that a tool should be given in a precise language, therefore being able to convey clear and vivid messages.

Step 7: Write down the desirable outcomes and competences

In this step, the creator is expected to write something similar to the content of the Step 2. For instance, if the General Aim-Objective of the specific tool was: 'students must learn to carry out quick and simple calculations based on the notions of addition, subtraction and multiplication', the Desirable Outcome that should be written here would be: 'after the activity's completion, students should have been familiarized with the processes of addition/subtraction and/or multiplication, thus being able to make simple mathematical operations, quickly and accurately.

Step 8: Think of a Method of Debriefing and Questions for the Evaluation of the Process

In order for the whole methodology to be comprehensive, it would be good to incorporate a Debriefing Section accompanied with some Questions for Evaluation, for both the educator and the target groups, thus also assessing the educational process itself. Some Sample Questions for Evaluation of the tool by the educator are being given below:

What skills has the target group gained within this tool? How will you apply the skills that the target group has gained during this process? What challenges did you encounter while applying the specific tool, regarding both the target group and the educational process? In which ways do the program goals meet your needs? What practices that you have been using up to today will be discontinued as a result of this tool? What new practices will you implement as a result of this tool? What was the result/impact of the target's group participation in terms of (i) the interest they had shown for the specific educational process; (ii) their engagement with the incorporated mathematical concepts; (iii) the supplementary objectives set by the educator.

Step 9: Select the Title

Although the title is the first thing that we encounter while reading a new tool for learning, the creator takes the final decision on what the title of the tool should be, slightly after completing the whole process of the tool's design and creation. In any way, try to select a catchy, attractive and imaginative title!

MATHEMATICS AND LEARNING DISORDERS

Math is a very concrete and exact subject. If you ask a child to tell you how much is 7 plus 3, the answer can't be approximated, it has to be very precise to give the right answer. You are either right, or either wrong. And usually, children don't get point for being almost right. So, more than any other object, math is causing anxiety because of the fear to be wrong, the fear of negative evaluation.

Because it is a "cumulative subject" (Brian Butterworth) knowledge is constructed so the new information is based and linked to the previous one. If you skip some content, the following one is less accessible. Making sustained progress in learning math is a very challenging process for people with Specific Learning Disorders.

The SLD are named Specific Learning Disorders because they are not the consequence of visual, hearing, or motor disabilities, nor mental retardation, emotional disturbance, or environmental, cultural, or economic disadvantage. They can affect cognitive development of one or more ability such as speaking, reading, writing, doing mathematic, plan and coordinate motor task.

THE SLD DO NOT HAVE A CAUSE DETERMINED BY:

- Physical disabilities
- Mental disability or developmental delays
- Psychological or sensory problems
- Socio-cultural factors

HERE IS A LIST OF THE SLD :

- Dyslexia - Difficulty with reading and spelling
- Dysgraphia - Difficulty with handwriting and some fine motor skills
- Dyscalculia - Difficulty with arithmetic and mathematics
- Dysphasia - Difficulty to produce and understand spoken language

ADDITIONALLY:

- Dyspraxia - difficulty with gross and fine motor coordination which is classified as Developmental Coordination Disorder and not as specific learning disorder but has an influence on students learning process

Even though many learners are affected by SLD, the estimation of their quantity differs. The European Dyslexia Association estimates that between 5 to 12 percent of the population have at least one SLD.

We already mentioned that for most people (including teachers, educators and decision-makers), mathematics is somewhat a complicated subject that can only be taught in a formal way. Many of us struggled with the queen of science and reasoning of the many abstract concepts but for students with SLD some of the tasks are very hard to overcome.

THE BIGGEST CHALLENGES IN MATHEMATICS FOR STUDENTS WITH SLD:

- making sense of numbers and how they work
- understanding the symbols and remembering the vocabulary
- understanding shapes: symmetry, relative size, their quantity and how to manipulate them
- weak long term and short-term memory which is necessary to automatize math procedures in calculus
- using drawing tools due to poor fine motor skills
- reading and organizational difficulties make it harder to solve multistep problems and tasks
- remembering multiplication tables that requires trying several approaches to find the most suitable

Making mathematics more “dys-friendly” starts by communicating with the student: getting to know what they like, how they approach the tasks and what discourage them during the process.

HERE ARE SOME TIPS FOR TEACHERS THAT COULD BE BENEFICIAL FOR ALL STUDENTS WITH SLD:

- using real objects that can be manipulated to explain geometry
- advise students to read problems aloud and help them to break the tasks in the smaller steps
- start a lesson with the outline of what is going to be learnt today and finish with a small recap of most important information’s
- increase comprehension by explaining recalling vocabulary and symbols in a form of math dictionary
- minimizing as much as possible the abstract aspect of mathematics by linking tasks with real life examples and applicability

- use books and photocopies with large print and big spaces between lines and paragraphs (line spacing of 1.5 is preferable). Font size should be 12-14 point.
- It is recommended to use a plain, evenly spaced sans serif font such as Arial and Comic Sans. Others: Verdana, Tahoma, Century Gothic and Trebuchet Remember that one size does not fit all and you should test it with your students to see what works best for them.

Using VR technology offers great opportunity to strengthen visualizing skills that are essential to learning mathematics. Algebra relies on a compressed system of written symbols with specific vocabulary, and it requires to automatize calculation tasks; whereas geometry relies on understanding shape, symmetry, relative sizes and quantities, how to manipulate them and how on trace them precisely on paper.

INTEGRATING VR TECHNOLOGY INTO THE NON-FORMAL APPROACH OF TEACHING MATHEMATICS

MODERN TECHNOLOGICAL INNOVATIONS THAT ARE CURRENTLY BEING USED FOR TEACHING NON-FORMAL MATHEMATICS

The United Nations Educational, Scientific and Cultural Organisation (UNESCO), a division of the United Nations, has made integrating ICT (Information and communication technology) into education part of its efforts to ensure equity and access to education. The following, taken directly from a UNESCO publication on educational ICT, explains the organization's position on the initiative.

UNESCO takes a holistic and comprehensive approach to promoting ICT in education. Access, inclusion and quality are among the main challenges they can address. The Organization's Intersectoral Platform for ICT in education focuses on these issues through the joint work of three of its sectors: Communication & Information, Education and Science]

In Europe, ICTs have been integrated and encouraged in schools for some years now. They promote non-formal education, especially for mathematics education.

SCREENS IN CLASSROOM: COMPUTER, TABLET, SMARTPHONE OR DIGITAL WHITEBOARD

Screens, in general, are part of the familiar environment of the students; it is a reassuring element that will give them confidence and encouragement to take initiatives. Students engage in active and motivated participation. Take the example of the digital whiteboard.

Source: Thomas Ricaud, Fermat Science

A touch interactive white screen is connected to a computer and a video projector. The interaction is done with an electronic pen or by simple contact, the screen transmits the information to the computer. Various software and accessories exist to see and manipulate notions of various fields of education.

Some of the benefits of using digital whiteboard, are that they:
- Give a possibility for the teacher to record the lesson, especially those steps that are more complex and then share it with students, so they can come back to it at needed moment;
- Promote student's participation and interactivity in class;
- Override motivation. By its fun, modern and new side (at first) it helps young people to maintain their interest and attention, which is particularly important for students with ADHD (Attention deficit hyperactivity disorder);
- Allows to represent some abstract entities (for example in geometry) to facilitate and consolidate the assimilation of mathematical concepts;

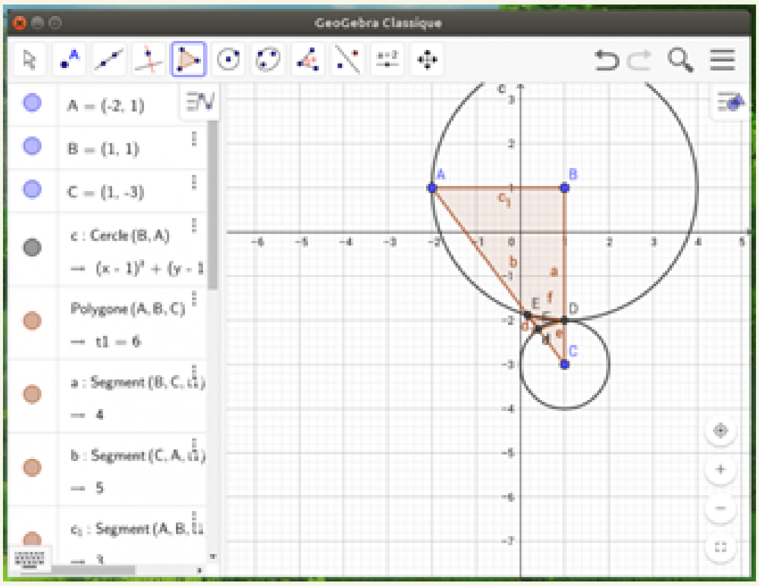
SOFTWARE AND APP FOR FACILITATING THE LEARNING OF MATHEMATICS

Nowadays, there are various free educational software to understand mathematics in a non-formal way. It is considered that, by mobilizing knowledge to solve the problems encountered, students will consolidate their skills and intellectual abilities. The practice of the game saves time in understanding, makes more durable essential skills in mathematics and allows them to develop several strategies.

Not only the technology itself is more "fun" than the simple paper-pencil technique, but the content of the software will allow these students to review concepts from a different point of view.

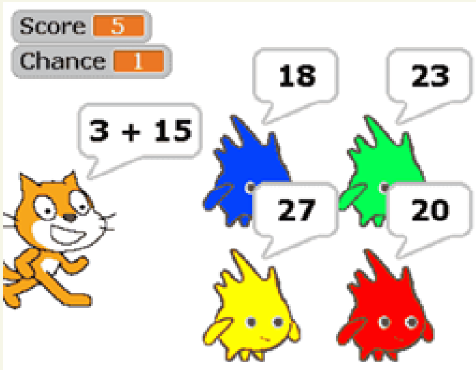
Some Examples

GeoGebra is a dynamic mathematics software for all levels of education that combines geometry, algebra, spreadsheet, graphex, statistics and infinitesimal calculus into one easy-to-use software.



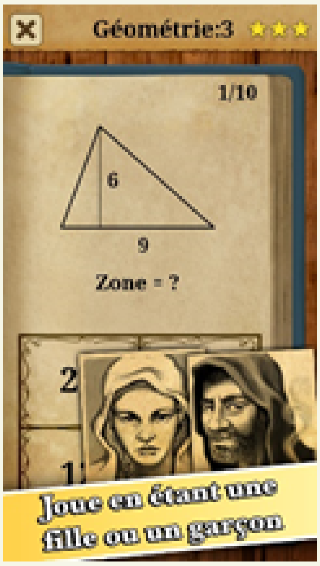
Source: screenshot from www.geogebra.org

Scratch is a visual programming software. This type of programming has the advantage of eradicating the possible errors of syntax, but it remains however a powerful programming language which makes it possible to use all the concepts of the algorithmic like the variables, the loops, the conditional instructions, the subprograms, etc.



Source: screenshots from https://scratch.mit.edu/

Le Roi des Maths is a fun math game. By marking yourself as a farmer, you create your character while answering math questions and improving your total score.



Source: screenshots from http://www.jeux.org/jeu/le-roi-des-maths.html

USE OF THE INTERNET AND SOCIAL NETWORKS TO APPROACH NEW CONCEPTS

Internet offers multiple means of support to the teaching of mathematics, allowing to diversify the way of teaching the notions and thus to multiply the possibilities of learning and assimilation.

Some Examples

Using mathematical routes in the city: **MathCityMap**
MathCityMap provides teachers with software that makes it easier to create routes to do mathematics. A downloadable application allows students to geolocate riddles and answer questions asked throughout the course. Immediate validation of their response allows them to start again or continue their journey. The enigmas proposed often come down to size (length, area, volume) but other fields are also possible.

Use of video (YouTube) support for courses: **Videos Scienticfiz (French)**
On the internet, and more particularly on YouTube, there are many videos talking about mathematics in a non-formal way. In France, there is an example of a school project: the YouTube channel Scienticfiz.

Source: https://www.youtube.com/channel/UCIUbSRKVVOpWI_xB3soLU1g/

Videos, written and filmed by the students, talk about math and magic, secret codes, mathematicians such as Fermat, Pythagoras...

Other youtubers who popularize mathematical concepts can also complement the traditional course:

NUMBERPHILE (ENGLISH) https://www.youtube.com/user/numberphile

MICMATHS (FRENCH) https://www.youtube.com/user/Micmaths

Sharing photos via Twitter and the hashtag: **#mathsenvie**

A NEW MODERN APPROACH: AUGMENTED REALITY

Students from the south of France gathered in 2018 to work on the design of the Jumathsji Mathematics Game. They created a game poster that uses augmented reality. That is to say that this poster (which can be posted online or displayed everywhere) will be able to talk, to discover through a smartphone a demonstration of the game, to read or hear a small biography for five mathematicians: Hypatia from Alexandria, Maria Agnesi, Euclid, Hipparchus and finally Cédric Villani. The questions are mathematical and written by the students. They relate to different fields.

Source: Pierre Henry, Casarotto Collège de Bazas

How does it work? By scanning the QRCODE present on the calculus, geometry, problems poster it becomes in the game: the valley of the numbers, the plain of 2D, the 3D spaces, a mountain of problem, there is even a part English Maths.

NEW PERSPECTIVES AND POSSIBILITIES THAT VR TECHNOLOGY COULD BRING TO MATHEMATICAL NON-FORMAL SCENARIOS

VR technology can bring a lot to the learning of mathematics. Indeed, as it was analysed in Chapter 1, the non-formal approach is beneficial for students' education in general, and particularly in math education.

Approaching maths differently

Teachers often use photos and videos to introduce a new concept with a non-formal approach. Students look at these elements, but most often they are in a passive position. On the other hand, VR allows students to dive directly into the lesson! Some examples that change the way to offer math lessons are:

In geometry, teachers use form hunting to bring this discipline to life: students must find all geometric shapes in their environment. They are often limited in shapes to find. With VR, students can travel around the world to search for shapes or circulate in a virtual geometric world.

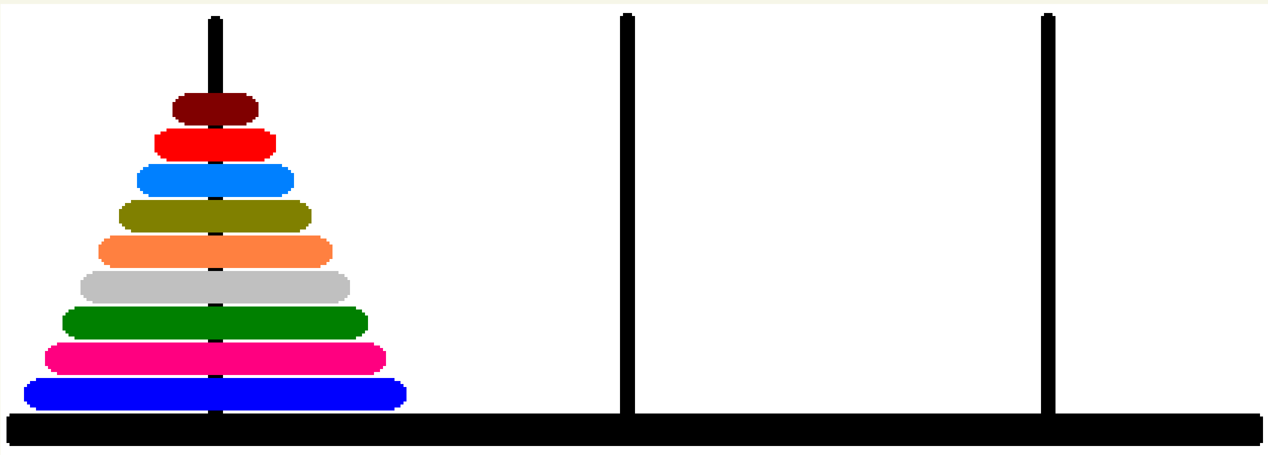
To work the mathematical logic, Escape Rooms are developing more and more in class but their installation, setting up a decor, an atmosphere still remains complicated. Imagine VR puzzles in the style of an Escape Room. The students would be much more motivated, and the choice of riddles is immense.

In solving a mathematical problem, students often know how to calculate, but have trouble in understanding how to find the result or deciding on an operation. Often this is attributed to a lack of mathematical vocabulary. By experiencing the problem in VR, the student can have a better understanding of it. For example, when the statement of the problem of water volume, tub to fill, or other, the student can test in VR fill containers, and find the solution by manipulating; impossible thing for most problems.

Puzzle Hanoi Tower

As seen in chapter 1, The Tower of Hanoi is a mathematical puzzle that allows approaching non-formally mathematical concepts. This puzzle can motivate students to work on these mathematical notions, but they often remain 'stuck' for manual resolution. Indeed, when the number of disks is large, it takes a lot of concentration to finish it.

Imagine this puzzle in VR! Students could test it, as in real life, but they could do a lot more such as:
- Ask for help and have clues
- To make mathematical notions appear directly in front of them
- Changing the colour of the discs for better understanding
- Listen to the legend around this puzzle



Source: www.france-ioi.org

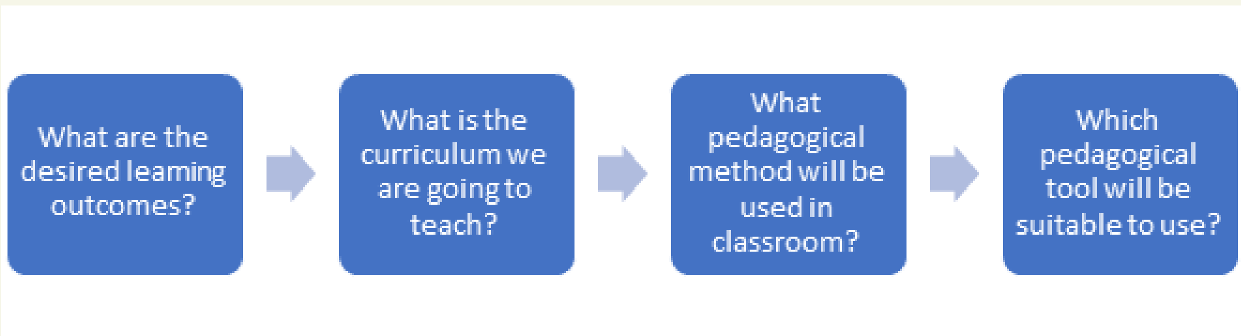
PEDAGOGICAL ASPECTS OF VR TECHNOLOGY

WHAT MAKES A VR MATH TOOL PEDAGOGICAL; PROPERTIES AND CRITERIA

In the past years we are all witnessing some interesting shift in the tools that teachers use in the classroom and to conclude pedagogical objectives of the subject that they teach. Motivated by the vast evidence that shows the effectiveness of active learning over traditional lecture in teaching STEM classes[1] many teachers search for various ways to shift their classroom practice, in order to incorporate active learning.

That transformation is visible in the continuous growth of EdTech industry. Globally, over $37.8 billion has been invested in educational technology companies between 1997 and 2017 and a striking 62% of that was invested in just the last three years between 2015 and 2017[2].

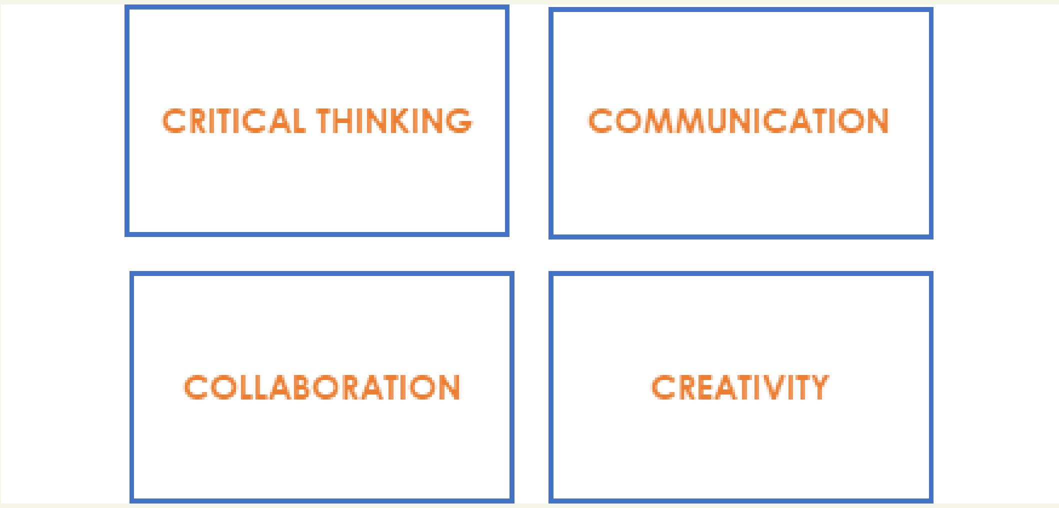
Good pedagogical design, traditional or digital, should provide an alignment between the curriculum we teach, the teaching methods we use, the learning environment we choose, and the assessment procedures we adopt (Biggs, 1999). Primarily, the role for teachers as expert designers of learning (Laurillard, 2013; Selwyn, 2016) is to establish learning tasks, supportive environments for learning, and conducive forms of social classroom relations. To follow this path, the following questions could be helpful:



[1] Freeman, S, Eddy, SL, McDonough, M, Smith, MK, Okoroafor, N, Jordt, H, Wenderoth, MP. (2014). Active learning increases student performance in science, engineering, and mathematics. Proceedings of the National Academy of Sciences, 111(23), 8410–8415.
[2] Metaari Research Report: The 2017 Global Learning Technology Investment Patterns

So, what criteria the VR tool should have to be called pedagogical?

In 2002, the partnership for 21st century skills, a coalition between the business community, education leaders and policy makers created the framework for 21st century learning. This framework presents a holistic view on modern teaching and learning practices. Within the context of core knowledge instruction, students must learn in 4Cs. These competences are considered to be essential for students to thrive in work and in life.



4Cs principle for 21st century skills

Digital tools can foster those skills in students and as Virtual Reality is being considered to have high potential in providing learning through experience, we will try to assess its pedagogical effectiveness through the 4Csprincipal:

Critical thinking refers to students' abilities to analyse, interpret, evaluate, make decisions and solve problems. It can be developed in classroom through guided inquiry and project or problem-based learning. So, for example during the lesson about amplitude of angles, students can use VR tool to analyse different building angles while 'visiting' the city of Florence and compare the use of mathematics in architecture in the past and in the present.

Clear **communication** focuses on expressing thoughts and opinions clearly to others. Deeper learning typically involves sharing what has been learned and interacting with others in a community. Leadbeater (2008) stresses that 'learning is best done with people rather than to or for them. It is more effective when learners are participants rather than merely recipients. Therefore, classroom activities that use VR tools should be based on relatable, realistic scenarios with elements of active communication between students. Innovative collaborative learning,

Innovative **collaborative** learning, challenges students to express and defend their positions, exchange different points of view, question others and seek clarification. It takes mutual respect, compromise, consensus building and shared responsibility. It can be done in project-based learning teams where peers through comparing results can
consider new uses for knowledge and develop new insights for future application. For example, lesson activities that use VR tools can be performed in pairs or small groups.

Creativity includes abilities such as brainstorming, refining ideas, being responsive to ideas from others and making ideas tangible and useful to others. Knowledge is not static therefore allowing a mix of acquisition and practices on project-based learning focusing on world-based challenges that can empower students to be active participants in the learning process. For example, a class might work together to brainstorm mathematical probability of an event to address environmental challenges while watching a 360 video about the impact of fossil fuels on the planet.

You can see from these examples that Virtual Reality, when used strategically, can really enhance how learners can apply classroom-gathered knowledge to real-world problems and take part in projects that require sustained engagement and collaboration (Barron and Darling-Hannibd 2008)

CONCLUSIONS/FINDINGS ON THE LEVEL OF USAGE AND THE ACADEMIC RESULTS OBTAINED FROM THE INCORPORATION OF VR TOOLS FOR MATHEMATICS IN NON-FORMAL AND FORMAL EDUCATIONAL SETTINGS.

In previous chapters we saw how some technological advances are keeping education and its tools relevant, Virtual Reality (VR) being one that is pushing its way into the mainstream as the hardware is becoming more affordable for school reality. In this section, some studies that show the impact of VR in education are being discussed:

Exploring the Use of VR Technologies in Mathematics Class
In 2017 a school in Hedmark County (Norway) conducted a six weeks pilot study involving four different classes of 3rd and 4th graders (34 students formed the experimental group) in mathematics, focusing on multiplication. The study’s purpose was to spot if the students' basic skills in mathematics will improve thanks to using Virtual Reality as a part of the learning activity. The process was monitored by Practice-Based Education Research Center (SEPU) at Innlandet University College The students took a mapping test in mathematics before and after the project.

A 5th grade math class at a different school (31 students formed the control group) that did not use VR technology and took the same mapping tests for comparison.

During the 6 weeks period, as a part of two teaching lessons, students used Oculus GearVR hardware and worked on tasks involving basic arithmetical operations. The experience was gamified as they could win points for correct answers. Teachers were provided with a platform to log into and keep track of the students' progression. This allowed to gather data on each students' progression over time, rather than just grading them. Creators from VR Education believe that viewing the student as someone who evolves, not only judges their performance, but it is also important for the education of the future.[1]

School	Gender	N	Total Cohen's d	Effect Cohen's d
Control group/school	Boys	13	.04	
	Girls	18	,27	
Intervention group/school (VR)	Boys	12	,53	0,49
	Girls	23	.23	-0,04

Source: www.vreducation.no/pdf/vr-maths-report-NO.pdf

This table shows a comparing development of boys and girls in both groups. An effect size of 0.49 in six weeks is considered a major effect/improvement. On the other hand, girls have showed no distinct improvement. During the test they expressed feeling nausea when using the VR glasses, therefore they used it less than boys.

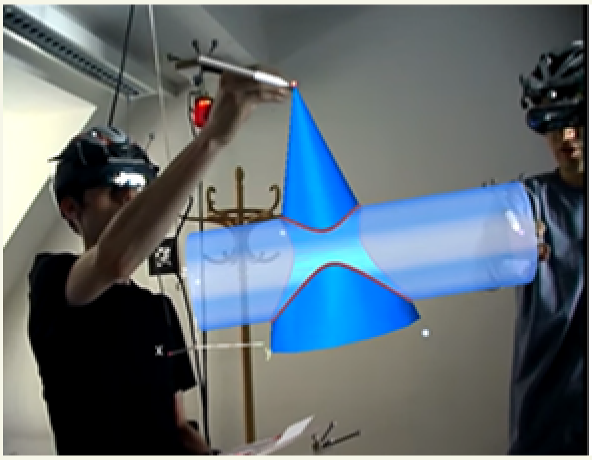
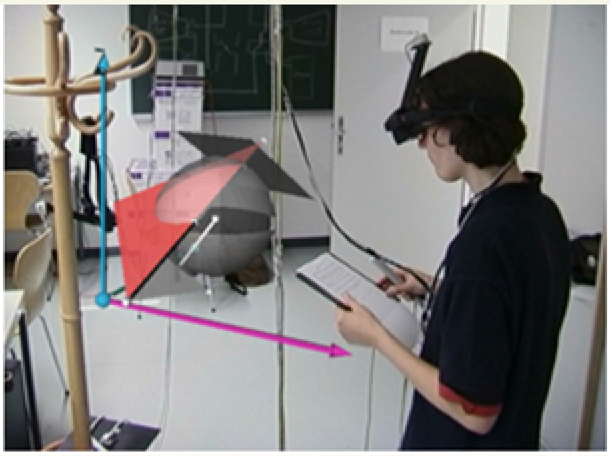
Construct3D: application for mathematics and geometry education at high school

In December 2008, Vienna University of Technology in Austria completed a research project called 'Construct3D' which is a construction tool in 3D in an immersive virtual environment to develop spatial abilities.

The aim of the tool was to create a simple and intuitive instrument with friendly user interface. Researchers integrated the collaborative augmented reality system Studierstube, which allows students to partially see and interact with the real world.

[1] https://www.youtube.com/watch?v=o6XIz_Afk9A

It includes an audio help system to give feedback and a feature to provide interaction with the teacher. The main areas of application in mathematics and geometry education are vector analysis, descriptive geometry and geometry in general. The tool provided students with a nearly tangible picture of complex three-dimensional objects and scenes.

Source: https://www.ims.tuwien.ac.at/projects/construct3d

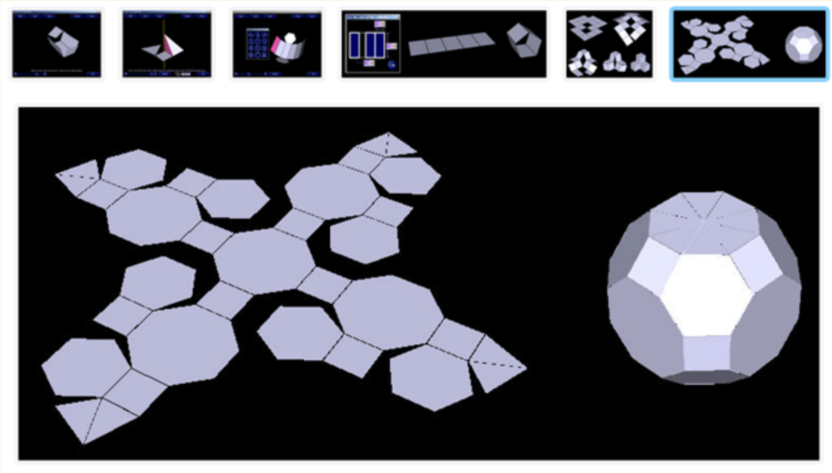
During the research an informal pilot study with 14 students in Vienna, aged 22-34 was conducted in order to evaluate the efficiency of the tool. Their task was to solve a construction example from vector analysis as thought in 10th grade in Austria with the help of a tutor. After the experience they had to complete a short survey about the overall experience and the tool.

The students were highly satisfied with their experience, wanted to use VR technology again and found it a great tool to explore math in playful settings. Several highlighted that the visualization of 3D shapes was much easier with VR than computer screens, however they needed the tutor's support to overcome some technical and physical (6 participants experience dizziness) difficulties. Nevertheless, they all imagined different use of Construct3D for solving simple problems in mathematics and geometry education.

The DALEST Project (Developing an Active Learning Environment for the learning of Stereometry)

The DALEST project was co-funded by the European Union under the Socrates Program, MINERVA, 2005 Selection. The work was done under an international collaboration of five universities University of Cyprus, University of Southampton, University of Lisbon, University of Sofia, University of Athens, N.K.M Netmasters and Cyprus Mathematics Teachers Association.

The applications developed were aiming in helping students design and create mathematical objects through pedagogical scenarios suitable for the teaching of stereometry and developing spatial thinking in elementary and middle schools. There were different levels and the students had to solve some math problems by measuring, cutting, (un)folding and manipulating the shapes to create mathematical objects.



Source: http://pavel.it.fmi.uni-sofia.bg/elica/dalest/on.html

Students who tested the applications and manipulate different types of nets, passed from 2D to 3D mode. All students have expressed satisfaction after their experience and at the same time, did not neglect the traditional work of creating paper models with paper and scissors. The general impression after testing the tool was that it provided more space for experimenting and playing with different ideas which students found exciting. (The tools are available on the project website: DALEST PROJECT).

To conclude, based on the examples provided above we can say that:

Even though, there are not many formally documented examples of using VR technology in mathematical education, the testimonials from researchers and students who used VR technology are very promising and its field worth exploring. With the optimistic opinions about using the technology come a valuable finding, emphasizing the fact that VR should be an addition to mathematics education and never a substitution

WHAT MAKES A VR MATH TOOL PEDAGOGICAL; PROPERTIES AND CRITERIA

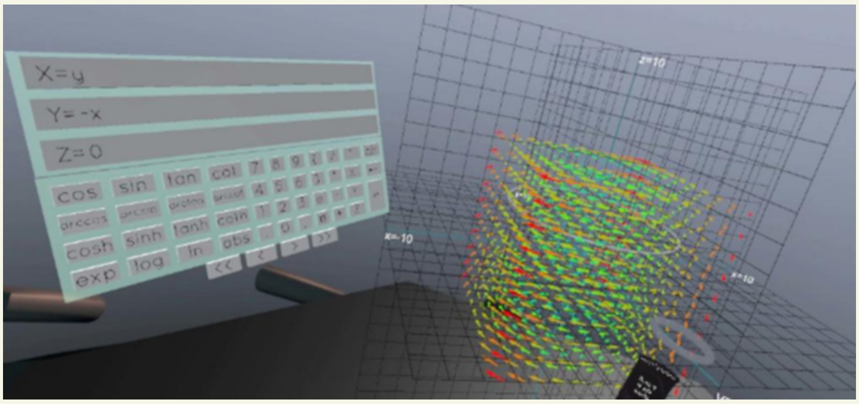
The VR content development is in its initial stages of its existence and the potential of applying the idea of immersive experience in education is attracting more stakeholders every year. Here are some of the existing propositions of VR Math tools;

CalcFlow

Calcflow is a free, open-sourced software developed by Nanome Inc. It requires an Oculus or Vive virtual reality headset to be used, whilst for the unexperienced users the developers have created an entire series of tutorial videos on YouTube.

Pedagogical objective: It allows studying and visualizations of vector calculus in an interactive environment. It can be used for mathematical modelling, making 3D Graphs manipulation while editing parameters on the go, creating your own parametrized function and vector field

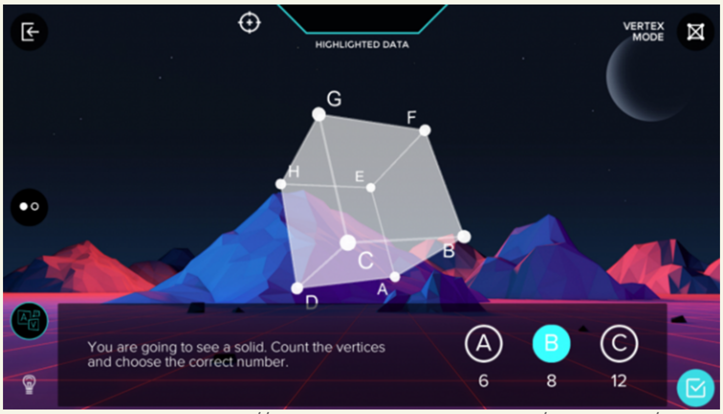
Target group: it is dedicated for older students in thelast classes of high school and it might be applied also in high education.



Source : https://vrroom.buzz/vr-news/trends/get-ready-love-math-vr-calculator

VR Math

VR Math is a part of VARP Edu, which is an educational platform for various subjects. It's an app that helps students to understand and educators to teach 3D Geometry. VR Math is a license-based service and subscription. The content itself is accessible through combination of phone and simple cardboard VR headsets or in 360 on a touch screen. At the moment (May 2019) it is still in beta version.



Source : http://kornelmeszaros.com/vr-math/

Pedagogical objective: It's mostly focused on geometry and students' development of spatial awareness and understanding. It has a range of interesting features including the ability to access content as a student at his/her own pace and promote self-learning.

Target group: secondary and elementary students

CalculusVR
This free app a passion project run by dr. Nicholas Long from Department of Mathematics and Statistics in Stephen F. Austin State University. Calculus allows user to visualize the concepts of multi-variable calculus in a virtual reality setting. In order to use it all you need is a Google Cardboard headset and a phone.

Source: https://longnesfa.wordpress.com/calculus-in-virtual-reality-project/

Pedagogical objective: Content is covering various modules like: 2D and 3D Coordinates and Graph, Curves and Surfaces, Vector Valued Functions of One Variable, Multivariable Functions. The user can specify their own objects for visualization as well go through lessons on the geometry and calculus of multi-variable functions and the corresponding surfaces.

Target group: University students

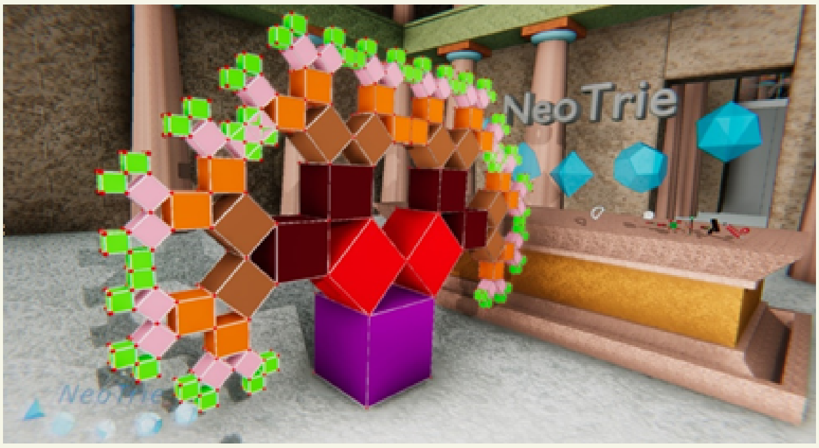
MashUp Math
MashUp Math is a group initiative of young educators who believe that *students learn math differently and that a one-size-fits-all approach is simply ineffective*. On their website www.mashupmath.com there are plenty of free resources like videos, worksheets and math puzzles. Their YouTube channel (in English) was last updated in 2017 but it has over 100 short videos which explain different mathematical challenges in an explicit way with a great focus on manipulations and non-formal approach.

Source: https://www.virtualiteach.com/single-post/2018/06/04/Maths-in-VR

Pedagogical objective: Teach mathematics in accessible and fun way.
The materials cover topics like: algebra, geometry, elementary and middle school math. MashUp Math started to experiment with 360 videos to stage algebraic problems. It is a simple idea that could be replicated easily in the classroom with a 360 camera.

Target group: students at age 8-15 years old

3D Geometry in Virtual Reality by NeoTireVR
This is a project run by José L. Rodríguez of Universidad de Almería. It's being under development in collaboration with Virtual Dor. The aim of the software is to enable users to create, manipulate, and interact with 3D geometrical objects and 3D models in general, of severaltypes.



Source: http://virtualdor.com/NeoTrie-VR/

Pedagogical objectives [1]: To examine aspects of plane geometry visible through the eyes of a third dimension; to introduce 3D geometry and modelling meant for 3D printing; to develop handicrafts and 3D visual skills; to stimulate deductive and inductive reasoning skills; highlight cooperative work and positive interdependence and motivate students by means of recreational, collaborative and competitive games.
Target group: As the project is in under a developmental phase the exact target group is not specified yet.

[1] http://virtualdor.com/NeoTrie-VR-Edu/

PRACTICALITIES OF USING COMBINED NON-FORMAL APPROACH AND VR TECHNOLOGICAL INNOVATIONS FOR MATHEMATICS IN THE CLASSROOM

THE PROCESS OF CONVERTING A MATHEMATICAL TOOL INTO A VR-SOLUTION AND IMPLEMENTING IT IN A LESSON SCENARIO

Introduction

When we use a mathematical model, we simplify a real-world situation, by making an abstraction: by doing so, we can explain, describe and predict the aspects of the real world through representations, i.e. interpretations of reality, such as diagrams, graphs, symbolic expressions. The problem is that students often see these representations as the end-product, and they are not able to perceive them as a tool to understand reality. Can this perspective be changed by using VR modelling?

There are two ways of solving mathematical problems using modelling:

Learning to model: students are asked to construct a model of the reality; to do so, each element of the model itself must be deeply understood. This is not easy, and it can be considered the goal of an educational process, rather than a way to teach new concepts.

Learning with models: this approach encourages students to solve problems by using existing models: students learn by changing parameters and by seeing the relationship among all the objects in the model. In order to use this approach, we need to create specific models, activities and manipulatives for every area of mathematics. These models can be easily created by Virtual Reality.

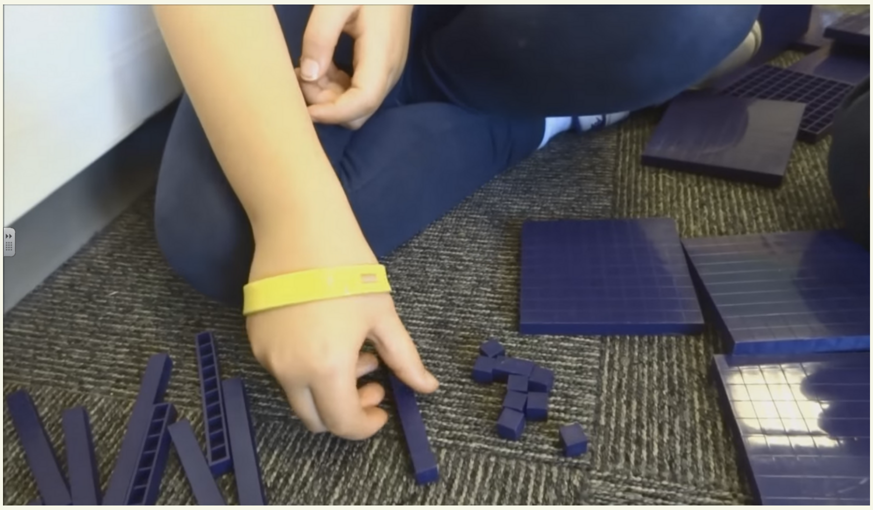
This section will help educators in selecting suitable topics and mathematical concepts to be taught by Virtual Reality applications in a non-formal scenario.

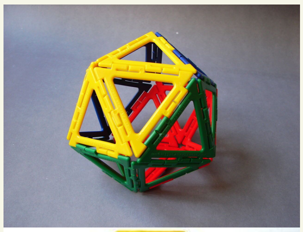
Step 1 – How to select topics and mathematical concepts

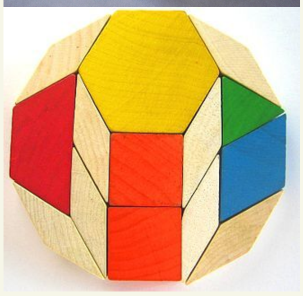
If teachers want students to learn mathematics by doing, they should select topics in which their students are asked to interact. In order to plan a successful non-formal lesson scenario, teachers should use tools which can let their students play an active role in learning. This can be done by using manipulatives.

The importance of manipulatives

One main concept a mathematics teacher should be familiar with is manipulatives: they are concrete models that involve mathematical concepts, i.e. real or virtual objects that can be used and moved around by students, such as algebra tiles, fraction pieces, pattern blocks, geometric solids, base-Ten blocks.







Sources:
base-ten blocks (left)-
https://en.wikipedia.org/wiki/Base_ten_blocks#/media/File:Dienes_blocks_used_by_a_8_year-old_student.png;
a Polydron icosahedron (top right) -
https://commons.wikimedia.org/wiki/File:Polydron_09489.jpg;
pattern blocks (bottom right). -
https://en.wikipedia.org/wiki/Pattern_Blocks#/media/File:Wooden_pattern_blocks_dodecagon.JPG

From a historical point of view, since the previous century some studies (Driscol 1983, Sowell 1989, Suydam 1986) had already showed that proper manipulatives can be beneficiary for students of any grade and ability, whenever they are used to explain and clarify a concept; for example, Suydam and Higgins (1976) had proved that lessons with manipulatives can improve mathematical skills and ability much more than lessons in which manipulative materials are not employed: this suggests that every student should have the chance to interact with manipulatives.
Their use must not be limited to the demonstration of the teacher, because they are meaningful and effective whenever they are used to involve students into interactive activities.

Step 2 – Which are the suitable topics and mathematical concepts

The use of Virtual Reality has great potential in mathematics and geometry education: it makes it possible to find innovative ways to teach mathematics, giving the chance to study more difficult and comprehensive problems, that were out of reach for students in earlier days.

Some of the most suitable topics, studied in most schools worldwide, to be taught through VR environments are:

- basic mathematical skills and/or concepts, such as four basic operations, fractions, mental calculation (for primary education)
- trigonometry, vector algebra, 3D geometry, graph visualizations and curve sketching, and other three-dimensional applications and problems (for secondary education)
- analysis (complex functions), linear algebra, differential calculus and differential geometry, projective geometry (for higher education/ high school).

In Chapter One, for instance, we had seen examples of pedagogical tools using non-formal mathematics. It has been addressed that the form can be very different. Some of these tools can be converted, and thus augmented, by using Virtual Reality.

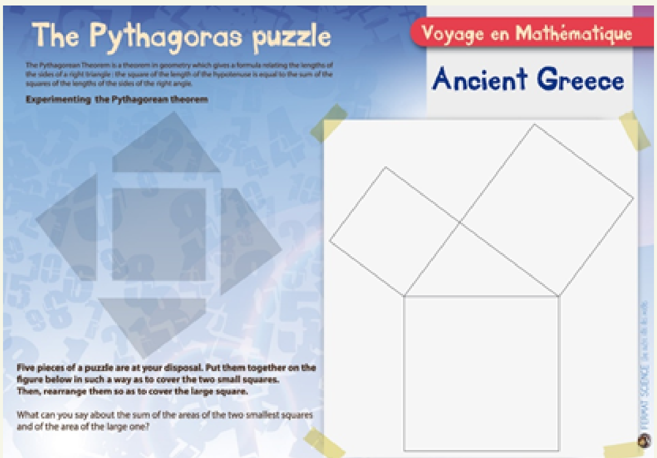
Let's take as an example the manipulation via a tiling of the proof of the Pythagorean theorem, and analyse its possible conversion into a VR hands-on solution.

The Pythagorean theorem is a property that calculates the length of the third side of a right triangle when the lengths of the other two sides are known.

Mathematical form:

In any right-angled triangle ABC it applies that $(BC)^2 = (AB)^2 + (AC)^2$

The goal is to manipulate wooden pieces in a puzzle tessellation game, to demonstrate the Pythagorean theorem.



Step 3 – How to embed the topic along with the math concept into a non-formal scenario

Virtual Reality must be used to adapt traditional mathematical content and can make it tangible, visible and understandable. Information received by vision are collected in a passive way: touch is an active and bi-directional process (Sourin & Wei 2009); it must help learners to understand complex, abstract, and non-intuitive concepts and systems, by manipulating relevant parameters and take on different points of view in real-time; it must also enhance learning by making the experience game-like: 3D graphics learning environments are engaging, interactive and flexible. Students familiar with videogames are highly motivated to play interactive games: this stimulates them to actively construct the knowledge through learning by doing.

The tool presented within Step 2 is a good example of embedding a math concept, in this case Pythagoras' Theorem into a non-formal scenario, inasmuch it allows many learning opportunities: in geometry (e.g characteristics of shape, vocabulary, paving), in history of mathematics (with the history of Ancient Greece etc). Moreover, it is linked to the curriculum of schools in Europe. The educational objectives are important. This educational tool works very well currently in math classes. Indeed, the non-formal approach of this theorem allows a better learning and a better comprehension.

The transposition into VR of "The Pythagoras Puzzle" can greatly increase the pedagogical possibilities compared to the existing tools. This is one of the reasons behind clearly defining the goal beforehand. In this case, do we want the student to know by heart Pythagoras' theorem? Or is it a first approach to the history of mathematics? Or is it both? This educational objective should be made explicit and clear from the beginning. It's also deemed necessary to define from the very beginning all the activities that could eventually result from the educational objectives.

For instance, when it comes to our example of the manipulation via a tiling of the proof of the Pythagorean theorem, in addition to the virtual manipulation that would bring the same benefits as in reality, the student could:

- solve other manipulations that demonstrate this theorem;
- prove the theorem mathematically and find its formula;
- access a fun video that demonstrates the Pythagorean theorem (example https://www.youtube.com/watch?v=YompsDlEdtc);
- build a Pythagorean tree (see the picture below);

- time oneself and challenge his/her friends;
- learn more about the mathematician Pythagoras by listening for example a soundtrack that tells his story or a video (https://www.youtube.com/watch?v=qmoxwZCiWEM&feature=youtu.be);
- see/study a bibliography containing works on this theorem or this mathematician, and even;
- meet the mathematician and ask him questions about his time and his theorem.

Step 4 - How to introduce innovative technologies and already existing VR Math applications into non-formal scenario?

Subsequently, we must choose whether the viewer (student) will be at the heart of the action or whether it will be placed according to other points of view. Will he interact with his environment or not? Will there be 360 ° videos or 3D modelling?

In any way, VR and 3D graphics learning systems should be integrated into a non-formal scenario in a game-like manner. In order to get the best results, VR math applications must satisfy these features according to Breuer (2011):

Interactivity: learning by doing and experimenting;

Multimedia-based: visualizing/preparing content and feedback by using 3D models, audio etc.;

Involvement: the game should be fully engaging to keep the player from distractions;

Challenge: increasing difficulty but beginner friendly, the game should always challenge the individual skills to keep players motivated;

Reward: rewards and feedback of progression should push self-efficacy and motivation;

Social Experience: providing communication channels to connect players.

Step 5 – Media, Techniques and Environmental context which could reinforce the educational process

Now, we should take final decisions concerning the environment context of our tool, in case that we are the creators/constructors of the VR-tool/app. The elements of decoration, the sound (lyrics, music) and the spatialization make it possible to give life to the scenario.

What will be most relevant for the educational objectives in our example: do we choose to recreate the Ancient Greece of Pythagoras, a playful world or a "geometric" world? Should music be classical for better concentration, paced to emphasize motivation or current to make repetitive tasks less restrictive and help the student to be more effective in problem solving? Will there be an "off" voice or will the interaction be via writing? One idea is to create various distinct environments for each of the exercises/tasks given within "The Pythagoras Puzzle".

In case that we are not targeting to the creation of a virtual tool/app from the scratch, we could alternatively work by focusing on readymade solutions for virtual manipulatives, specifically developed for the topics-concepts we want to teach: there are a few commercially available software tools usable for learning and teaching, such as:

Mathcad (https://www.ptc.com/en/);
Maple (https://www.maplesoft.com);
Mathematica (http://www.wolfram.com/mathematica/);
MATLAB (https://uk.mathworks.com/products/matlab.html);
Geometer's Sketchpad (http://www.dynamicgeometry.com/).

These tools, give students the possibility to see images, but without having the experience of getting immersed within the 3D scene: this lack of involvement can result in a lesser quality learning experience, but it can be still a valid complementary tool to be used in order to strengthen students' mathematical skills.

Step 6 – Creating the experience

Once the architecture of the future app has been defined, all that remains is to create the various elements that will compose "The Pythagoras Puzzle" VR or any other hands-on scenario which is deemed suitable to be converted into a VR tool.

A PRACTICAL GUIDE ON HOW TO CREATE A PROGRESS TIMELINE

To use VR in the classroom, you need at least two things: a smartphone that can download/run apps or video files, and a headset. Android Phones with Gyroscope and Magnetometer sensors make a smartphone compatible for VR. Most of the mid to high-end devices do come with a gyroscope and a magnetometer making them "The Perfect" companion for VR/AR. As a software you can use free or paid apps or even movies made with 360 °cameras. Even though these movies don't offer the same 3-D experience as VR apps, they're still amazingly interactive and immersive when viewed through headsets.

The observations presented below are made taking into account a class of about 30 students and a course duration of 50 minutes. The lesson plan/teaching project reflects the way in which the teaching activity will be oriented, in order to achieve the objectives. The lesson is carried out in a sequence of stages, which gives it a certain structure. This structure isn't compulsory nor rigid. The type of lesson shall be determined according to the general objective of the lesson. The main types of lessons are as follows:

The mixed/ combined lesson –aims to achieve approximately equal extent of several teaching tasks (communication, systematization, fixation, verification), being the most common type of lesson encountered in didactic practice, especially at general school level.

The lesson of communication/appropriation of new knowledge – has the fundamental objective of acquiring new knowledge and developing intellectual capacities and attitudes; Thus, the acquisition of the new, the other stages corresponding to the mixed type (different to the communication/appropriation of new knowledge) being present, but with a much less importance, depending on the age of the students (in high-school, the communication lesson tends to have a mono-stadial structure);

The lesson of developing skills and abilities (specific to mathematics) seeks to familiarize the students with different intellectual work methods, to get them used to organizing and conducting an independent work, to the practical application of knowledge.

The three lesson types presented have about the same general structure, but their difference lies in the importance of their stages- presentation of the content vs. knowledge fixation.

The structure and timeframe of a mixed/ combined lesson:

	Lesson stages	Duration	VR
1	Organizational moment	3 min	
2	Updating/checking the previously acquired knowledge and skills, checking the homework	10 min	
3	Preparing for the new topic	3 min	
4	Announcement of the new topic and lesson objectives	3 min	
5	Optimal presentation of the content and directing learning by various ways depending on its nature; Training students in solving various tasks, gradually introduced, depending on the level of difficulty, on the psychological components involved in the learning process; differentiated and individualised activities, reported to the psychological particularities regarding age and individual.	15 min	Yes
6	Fixation of knowledge through repetition, systematization of knowledge and skills, through applications involving transfer of skills and abilities in contexts different from those created during the stage of orienting the learning process.	10 min	Yes
7	Ensuring retention and transfer by specifying the homework, accompanied by the explanations needed to continue learning and to ensure the operation with new knowledge and skills in new conditions/contexts	4 min	
8	Appreciations and recommendations	2 min	

VR apps and 3-D movies being more attractive and interactive can be used when presenting the new content to observe and explore the links between mathematical theory and the surrounding reality. There are (or can be designed) interactive applications to be used at the time of knowledge fixation through exercises and problem solving.

Other types of lessons are:

The lesson of fixation of knowledge and of developing skills and abilities – aims, in particular, the consolidation of the knowledge acquired and the completion of students' gaps; this type of lesson becomes effective if it resizes the content around ideas with relevant cognitive value, so students become able to make connections that allow them more complex and operative applications;

The general structure and duration of this type of lesson is shown in the following table:

	Lesson stages	Duration	VR
1	Organizational moment	3 min	
2	Announcement of the topic and objectives by presenting the revision plan, established and communicated to the students in the previous lesson	5 min	
3	Orienting the learning process by directing the revision/consolidation process: systematization, deepening knowledge and/or skills, establishing new correlation between them; transfer in new instructional contexts based on solving various tasks with progressive degrees of difficulty; synthesis, individual or group work; differentiated and individualised activities.	30 min	Yes
4	Ensuring retention and transfer by specifying the homework, accompanied by the explanations needed to continue learning and to ensure the operation with new knowledge and skills in new conditions/contexts	10 min	
5	Appreciations and recommendations	2 min	

The lesson of verification and appreciation of the school results– aims mainly to establish the level of training of the students, but also the framing of their knowledge in new reference frameworks with a role in the future learning paths.

The general structure and duration of this type of lesson is as follows:

	Lesson stages	Duration	VR
1	Organizational moment	3 min	
2	Announcing the assessment objectives/competences to be evaluated – the proposed topic for the evaluation and how the evaluation precess will be carried out ; in the case of a large content evaluation, the topic will be established and announced in advance.	5 min	
3	Performance Assessment – This step is correlated with the specifics of the method (generally written, oral or practical evaluation) and the evaluation tools; the students are informed about the scales/and/or the assessment criteria.	30 min	Yes
4	Ensuring the inverse connection by checking the results directly/individualy; highlighting the typical mistakes, accompanied by additional explanations in order to clarify them.	10 min	Yes
5	Appreciations and recommendations	2 min	

As far as the assessment, VR applications can be designed to contain tests with different types of objective (dual choice, multiple-choice, or pairing) or semiobiective (with short answers or add-in answers).
Among their advantages is the fact that the pupil receives the correct answer on the spot, thus achieving the inverse connection immediately. Another advantage is the possibility to keep/save students' answers and to make them available to the teacher in a database.

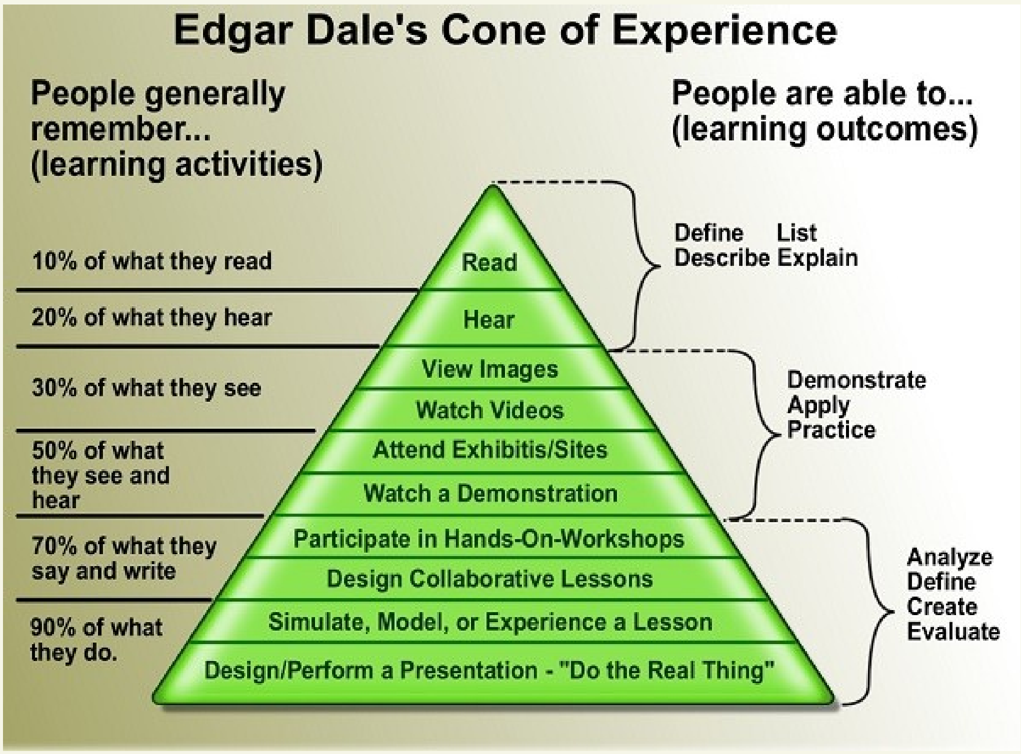
Some of the disadvantages of objective/semiobiective items are that they do not allow the evaluation of complex learning objectives such as the originality and creativity of the students, their abilities to organize and integrate their ideas, the interpretation and application of the information by them acquired.

As can be seen, the realisation of a lesson project depends on a number of variables such as the nature of the content, the objectives pursued, the level of training of the students and the type ofteaching strategies used.

A PRACTICAL GUIDE ON HOW TO INTEGRATE THESE LESSONS AND VIRTUAL REALITY TO CURRENT CURRICULA IN THE PARTNER COUNTRIES.

INTRODUCTION AND PRESENTATION OF VIRTUAL REALITY INTO MATHS LESSONS

Research has shown that we remember only 10% of what we see but 90% of what we experience (Image 1).



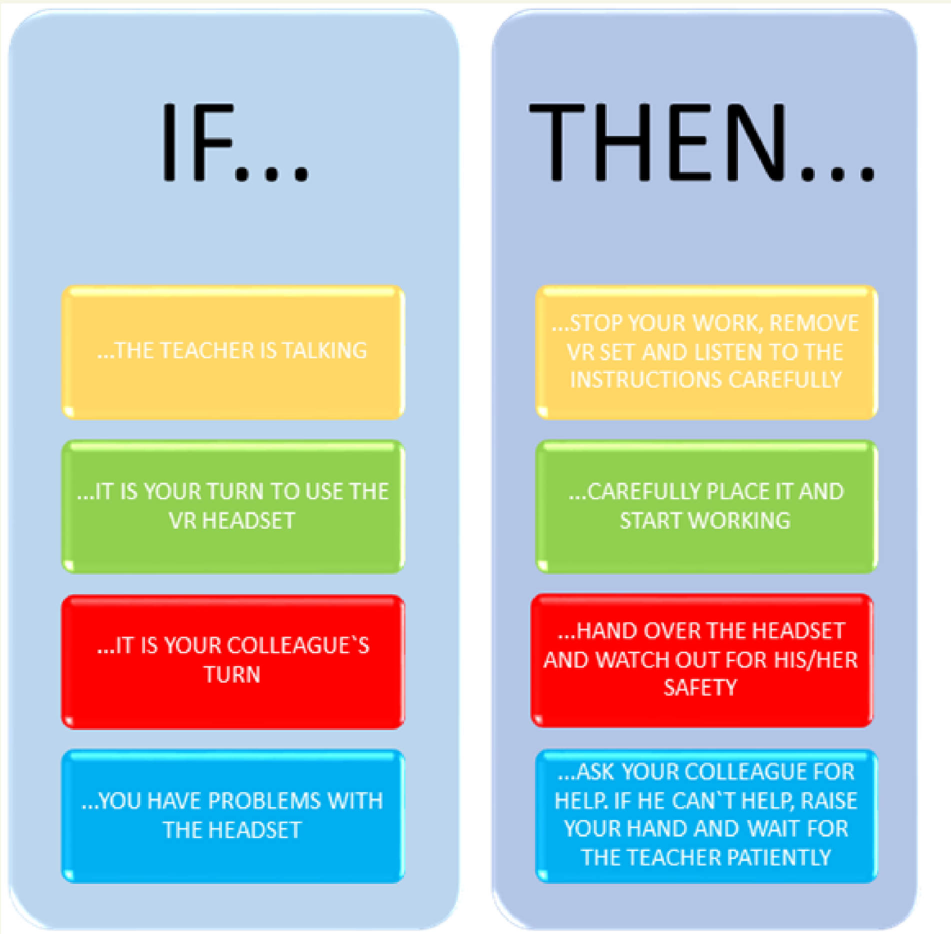
Source: Image 1 https://www.td.org/Publications/Blogs/Science-of-Learning-blog/2015/03/Debunk-This-People-Remember-10-Percent-of-What-They-Read

Technology will never replace the teacher, but it can help in developing students` skills and competencies needed for life in the 21st century – communication, creativity, collaboration and critical thinking.

Virtual reality in learning is transforming boring contents and making them interesting, so even less active students are becoming engaged. Creating scenarios for learning by using virtual reality provides a controlled learning environment, therefore virtual experiences created through virtual reality allow students to experience scenarios without getting exposed to certain risks. So, VR technology enables teachers to bring their students to enjoy and safely learn from experience.

Virtual reality provides student with possibilities to make mistakes and to learn from these mistakes without consequences, giving them a chance to try again and achieve success - practice and consolidate.

Before the first use of VR technology by the students in a classroom it is recommended to take time and discuss the use of VR in classroom, describe the experience and ask about their expectations. It is the teachers' responsibility to create the right atmosphere in the classroom and set goals that are supposed to be accomplished.
After the first students` VR experience, the teacher should discuss this experience with them, asking them what they saw, discuss their feelings, impacts and outcomes.
Using virtual reality in the classroom will probably bring joy and excitement to students. It can affect students' learning perception. Therefore, teachers must set clear rules and give detailed instructions for learning in virtual environment and students must be aware of the rules of conduct.
The teacher must emphasize on the importance of good communication between the students as well as between students and teachers during VR usage.



When introducing the topic and applying VR, the teacher must focus students` attention to the content itself, so they don`t miss out any details which are important for learning. Teenagers are prone to emotionally experience and can get carried away when using VR. That is why the teacher must stress the importance of outcomes in both real and virtual world. Without a high level of focus, students could miss out on details that could be important in their VR learning experience.

Rules and safety when using VR in the classroom

In order to achieve maximum efficiency of a lesson when using the new technology, it is necessary to define the work methods and rules of conduct in and outside the classroom in advance. Before applying the technology itself, students must be introduced to safety precautions, which can be summarised into the following points. It is necessary to minimise all factors of external disturbance for the classroom where VR is being used. Clear warnings on the doors should keep other people from coming in and disturbing the work. It is also important to enable students to work in silence. When using headsets for virtual reality it is impossible for students to see their surroundings. Therefore, any noise or disturbance must be minimised. Any physical obstacles must be removed. Rules of conduct and communication during the work must be clearly visible.

- Listen to the teacher carefully
- Remove physical obstacles before using VR
- Always work in pair - never alone
- Keep the device clean

Debriefing and Evaluation: How could the educator discuss this innovative educational process with the students

Teachers can use different types and methods of assessment to check if the outcomes of a lesson have been achieved. Some templates of evaluation methods that can be used are presented below

1. METHODS OF ASSESSMENT

- Analysing portfolios
- Observing students' performance in activities, practical work and research
- Oral examination
- Analysing students' reports, posters, mind maps and research
- Evaluation of a discussion in which a student participates
- Written examination

2. LESSON EVALUATION

Evaluation of statements on scale from 1 to 5, 1 being not at all, and 5 being completely yes.

1. I like the way of work in this lesson.	1	2	3	4	5
2. This lesson was interesting.	1	2	3	4	5
3. It is clear what I was supposed to learn in this lesson.	1	2	3	4	5
4. The subject matter was clearly explained.	1	2	3	4	5
5. I have learned the subject matter.	1	2	3	4	5
6. I think I actively participated in this lesson.	1	2	3	4	5
7. I was more active in this lesson than usually.	1	2	3	4	5
8. By being active I contributed to the quality of the lesson.	1	2	3	4	5
9. I was motivated for work in this lesson.	1	2	3	4	5
10. I prefer using VR in lessons.	1	2	3	4	5
11. Name two things you liked in this lesson.					
12. Name two things you didn`t like in this lesson.					

3. TEAM-WORK EVALUATION

ELEMENTS	YES	PARTLY	NO
1. We have completed the task successfully.			
2. Each member of the team has contributed to the fullest.			
3. All members of the team have participated in completing the task.			
4. We have accepted each other`s opinions.			
5. I like this way of learning.			
6. I can explain what I have learned after this lesson.			

BIBLIOGRAPHY

- Buldioski, G., Grimaldi, C., Mitter, S., Titley, G., & Wagner, G. (2002). Training Essentials. T-Kits (Vol. 6).Strasbourg: Council of Europe Publishing. Retrieved from http://youth-partnership-eu.coe.int/youth-partnership/documents/Publications/T_kits/6/tkit6.pdf
- Carver, Rebecca L. (1996). Theory for practice: A framework for thinking about experiential education. The Journal of Experiential Education. 19:8-13.
- Carver, Rebecca L. (1998). Experiential education for youth development. Youth Development Focus, 4-H Center for Youth Development, University of California, Davis, fall.
- Catterall J., Chapleau R., and Iwanaga J. (1999). Involvement in the arts and human development: General involvement and intensive involvement in music and theatre arts. In E. B. Fiske (ed.) Champions of Change (pp. 1-18). Washington, DC: Arts Education Partnership.
- Chisholm, L. (2001). Towards a revitalisation of non-formal learning for a changing Europe. Report of the Council of Europe Youth Directorate Symposium on Non-Formal Education. Strasbourg, 13 – 15 October 2000.
- Dewey, John. (1916). “Democracy and education: an introduction to the philosophy of education.” New York: The Macmillan Company.
- Enfield, Richard. (2001). Connections between 4-H and John Dewey’s philosophy of education. Youth Development Focus, 4-H Center for Youth Development, University of California, Davis, winter.
- Enfield, Richard. (2000). SLO Scientists: Families having fun with Science Clubs. In Braverman, Marc T., Ramona M. Carlos, and Sally M. Stanley, Eds. Advances in Youth Development Programming: Reviews and Case Studies from the University of California. Oakland, CA: University of California Agriculture and Natural Resources.
- Gillert, A., Haji-Kella, M., Jesus Cascao Guedes, M. de, Raykova, A., Schachinger, C., & Taylor, M. (2000). Intercultural Learning. T-Kits (Vol. 4). Strasbourg: Council of Europe Publishing. Retrieved from http://youth-partnershipeu.coe.int/youthpartnership/documents/Publications/T_kits/4/tkit4.pdf
- Johnson C. M. and Memmott J. E. (2006). Examination of relationships between participation in school music programs of differing quality and standardized test results. Journal of Research in MusicEducation, 54, pp. 293-307.
- Russell, Stephen T., and Glen H. Elder, Jr. (1997). “Academic Success in Rural America: Family Background and Community Integration.” Childhood. 4:169-181.
- Van Horn, Beth E., Constance A. Flanagan,and Joan S. Thomson. (1998). The First Fifty Years of the 4-H Program (Part 1). Journal of Extension, 36(3): http://www.joe.org/joe/1998december/comm2.html.
- Siurala, Lasse (2008): The variety and differences amongst the concepts of nonformal education. Vilnius, 16.04.2008.
- Walker, Joyce. (1998). “Youth Development Education: Supports and Opportunities for Young People.” The Center. University of Minnesota Center for 4-H Youth Development. Winter:10-13.
- Bell, J. T., & Fogler, S. H. (1995). The investigation and application of virtual reality as an educational tool. Proceedings of the American Society for Engineering Education Annual Conference, Anaheim, CA;
- Breuer, J. (2011). Spielend lernen? Eine Bestandsaufnahme zum (Digital) Game-Based Learning. Landesanstalt für Medien NRW;
- Clements, D. H., & McMillen, S. (1996). Rethinking Concrete Manipulatives. Teaching Children Mathematics, 2(5), pp. 77 and 270-279;
- Dorward, J., & Heal, R. (1999). National Library of Virtual Manipulatives for Elementary and Middle Level Mathematics. Proceedings of WebNet99 World Conference on the WWW and Internet, pp. 1510-1512. Honolulu, Hawaii Association for the Advancement of Computing in Education;
- Driscoll, Mark J. (1983). Research within Reach: Elementary School Mathematics and Reading. St. Louis: CEMREL;
- Durlach, N., Allen, G., Darken, R., Garnett, R. L., Loomis, J., Templeman, J., & von Wiegand, T. E. (2000). Virtual environments and the enhancement of spatial behavior: Towards a comprehensive research agenda. Presence -Teleoperators and Virtual Environments, 9(6), 593-615;
- Fennema, E. H. (1972). Models and mathematics. The Arithmetic Teacher, 19, 635-640. Retrieved 28 March, 2019, from https://www.jstor.org/stable/41188128
- Hiebert, J., & Wearne, D. (1992). Links between teaching and learning placevalue with understanding in first grade. Journal for research in Mathematics Education, 23, 98-122. Retrieved March 28, 2019 from https://www.jstor.org/stable/749496 ;
- Lesh, R. A. (1979). Applied ProblemSolving in Early Mathematics Learning. Northwestern University;
- Moyer, P. S., Bolyard, J.J., & Spikell,M.A. (2002). What are virtualmanipulatives? Teaching Children Mathematics, 8(6), 372-377. Retrieved 21March, 2019, from https://www.jstor.org/stable/41197834;
- NCTM, National Council of Teachers ofMathematics. (1989). Curriculum and evaluation standards for schoolmathematics. Reston, VA, p. 17;
- Post, T. (1981). The Role of Manipulative Materials in the Learning of MathematicalConcepts. In Selected Issues in Mathematics Education. Berkeley, CA:National Society for the Study of Education and National Council of Teachers ofMathematics, McCutchan Publishing Corporation;
- Rizzo, A. A., Buckwalter, J. G., Neumann,U., Kesselman, C., Thiebaux, M., Larson, P., & Van Rooyen, A. (1998). The virtual reality mental rotation spatial skills project. CyberPsychology andBehavior, 1(2), 113-120;
- Sourin, A., Lei, W., (2009). Visual immersive haptic mathematics,Virtual Reality (2009) 13, pp. 221–234;
- Sowell, Evelyn J. (1989). Effects of Manipulative Materials in Mathematics Instruction. Journal for Research in Mathematics Education, 20:498-505. Retrieved March 25, 2019, from https://www.jstor.org/stable/749423;
- Suydam, M. N. (1986). Research Report: Manipulative Materials and Achievement. ArithmeticTeacher, 33, pp 10, 32. Retrieved 25 March, 2019, from https://www.jstor.org/stable/41192833 ;
- Suydam, M. N., & Higgins, J. L. (1976). Review and Synthesis of Studies ofActivity-Based Approaches to Mathematics Teaching. Final Report, NIEContract No. 400-75-0063;
- Taxén, G., Naeve, A. (2001). CyberMath: ASystem for Exploring Open Issues inVR-based Education, Center for User Oriented ITDesign, Royal Institute of Technology, Stockholm; Winn, W., & Bricken, W. (1992). Designing virtual worlds for use inmathematics education: The example of experiential algebra. EducationalTechnology, 32(12), 12-19. Retrieved 22 March, 2019, from https://www.jstor.org/stable/44425562
- https://www.weareteachers.com/virtual-reality-classroom/https://www.researchgate.net/publication/325248053_Learning_in_a_virtual_environment_Implementation_and_evaluation_of_a_VR_math-gamehttp://www.isj-db.ro/static/files/curriculum/Informatica_TIC/Anexa_2_Tipuri_de_lecii_FINAL.docxhttp://www.isjcs.ro/Definitivat/GHID_DEF_pentru%20site.pdf
- http://dppd.ulbsibiu.ro/ro/cadre_didactice/adriana_nicu/cursuri/Pedagogie%202_curs_8_Testul%20docimologic.pdfSources of sub section 4.3www.cfeduex.com
- https://www.virtualiteach.com/single-post/2018/06/04/Maths-in-VRhttps://nearpod.com/blog/virtual-reality-math/
- http://blog.scientix.eu/2018/05/the-use-of-immersive-virtual-reality-in-the-mathematics-classroom/
- https://www.veative.com/deployment/classroom-usage https://files.eric.ed.gov/fulltext/EJ1131313.pdfhttps://www.cfeduex.com/setting-a-conducive-atmosphere-in-class-through-vr-learning-etiquette/
- https://www.cfeduex.com/emphasizing-the-benefits-of-virtual-reality-in-universities-and-its-students/
- https://observatory.tec.mx/edu-trends-augmented-and-virtual-reality/
- https://www.pinterest.com/https://loomen.carnet.hr/
- https://www.google.com/search?q=confucius+quotes+i+hear+i+forget&tbm=isch&source=univ&client=firefox-b-d&sa=X&ved=2ahUKEwi6g6vhiZbiAhUqtIsKHfQgD64QsAR6BAgJEAE&biw=1366&bih=645#imgrc=m18cStENO6xstM:

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