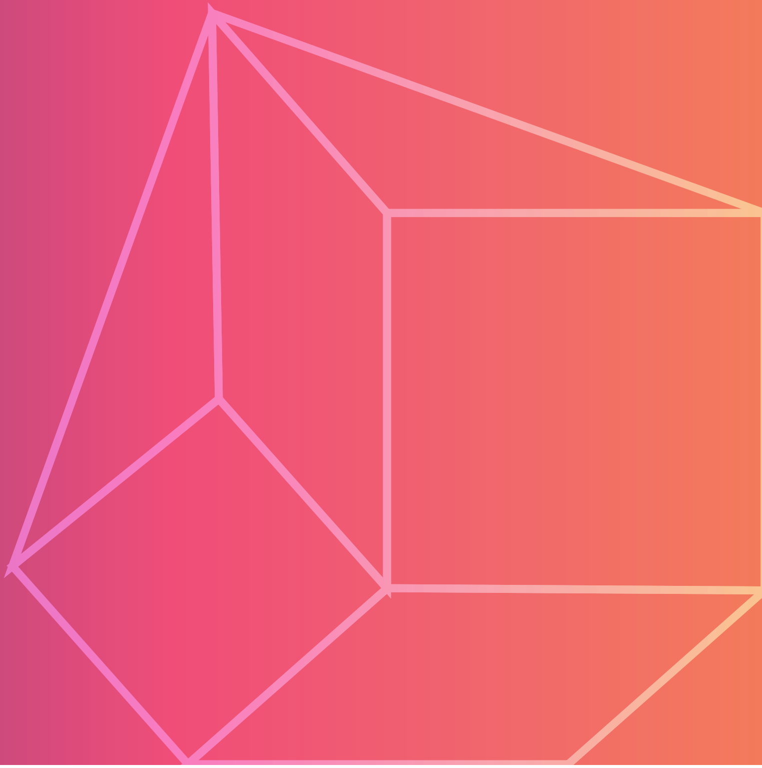
02



Cosine Rule

Topic: Trigonometry
Theme: Illustrate the Cosine rule by cutting and pasting
Abilities: Visualize a trigonometric theorem. Demonstrate it by manipulating figures
Material: Cardstock; triangle rulers; pencil; colored pens; scissors.
Level: Age 17/18

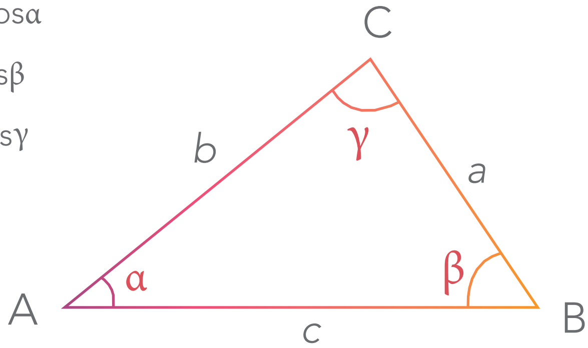
Math Reality

The Cosine Rule

(also called law of cosines) relates the lengths of the sides of a triangle to the cosine of any one of its angles.

Following the notation in the figure, the Cosine Rule can be expressed as:

- $a^2 = b^2 + c^2 - 2bc \cdot \cos\alpha$
- $b^2 = a^2 + c^2 - 2ac \cdot \cos\beta$
- $c^2 = a^2 + b^2 - 2ab \cdot \cos\gamma$



This rule was already known in the 3rd century B.C. since Euclid's *Elements* contain an equivalent version of this statement, though the notion of the cosine was not yet developed at that time. Euclid treated separately the cases of obtuse triangles and acute triangles (corresponding to the two cases of negative or positive cosine).

In the 15th century, Jamshīd al-Kāshī, a Persian mathematician and astronomer, provided the first explicit statement of the law of cosines in a form suitable for modern usage. That is why, since the 1990s, in France the law of cosines is still referred to as the Théorème d'al-Kāshī.

In Italy the law of cosines is usually known as Teorema di Carnot, and it is useful, along with the sine law, to solve any triangle.

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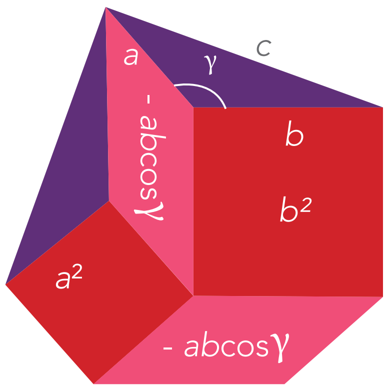
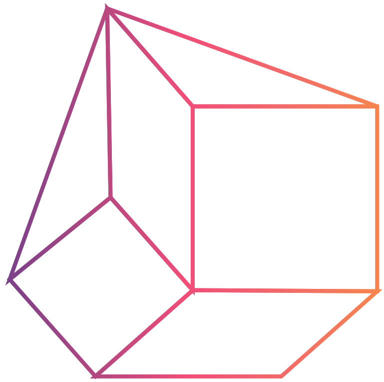
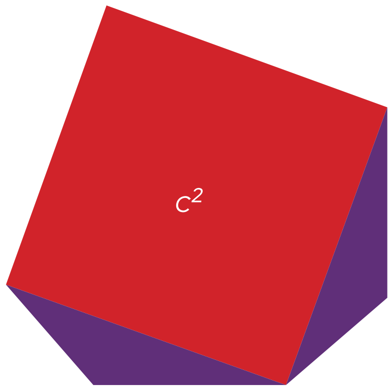
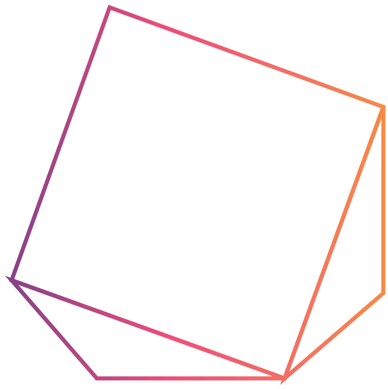








1) Draw a triangle: the violet one in figure with sides a, b, c and an obtuse angle γ.

2) With triangle rulers draw in the first figure:

- draw the big red square with side b;
- draw the pink parallelogram by adding the two missing parallel sides;
- draw the little red square with side a;
- draw the second violet triangle by adding the third missing side;
- draw the lower pink parallelogram by adding the two missing parallel sides.

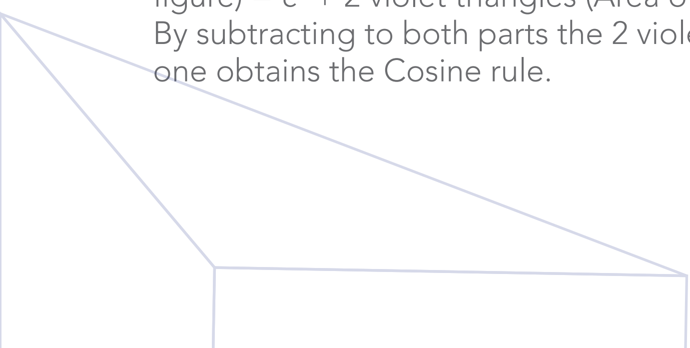
3) Find the area of the pink part of the figure using basic trigonometry: the pink figure is a parallelogram, whose base is b. The height of this parallelogram is $-a \cdot \cos\gamma$. The area is base times height, and it equals $-ab\cos\gamma$.

4) With triangle rulers draw the whole second figure:

- draw the big red square with side c;
- add two violet triangles as shown in the figure.

5) Cut the two figures and verify the equivalence of their areas. As one can easily see, the two figures are superposable, thus have the same area. It means that:

$a^2 + b^2 - 2ab\cos\gamma$ + 2 violet triangles (Area of the left figure) = c^2 + 2 violet triangles (Area of the right figure)
By subtracting to both parts the 2 violet triangles, one obtains the Cosine rule.



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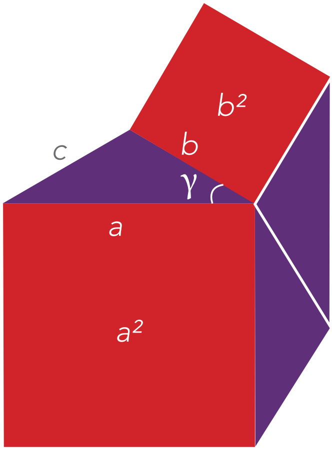

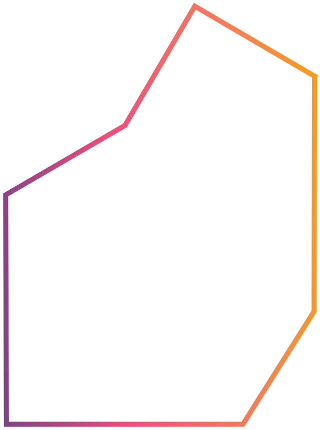

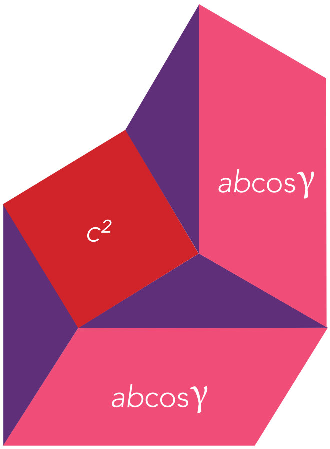

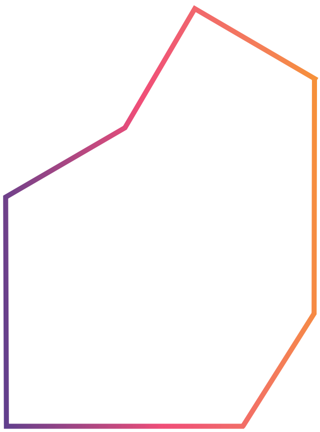

Same procedure, but with an acute angle triangle.

1) Draw a triangle: the violet one in the figure with sides *a*, *b*, *c* and an acute angle γ.

2) With triangle rulers draw in the first figure:

- draw the big red square with side *a*;
- draw the little red square with side *b*;
- complete the figure on the right by drawing a parallelogram and divide it in two violet triangles;

3) With triangle rulers draw in the first figure:

- draw the little red square with side *c*;
- add three violet triangles as shown in figure;
- complete the figure by adding two pink parallelograms as shown in figure.

4) Find the area of the pink part of the figure using basic trigonometry: The pink figure is a parallelogram, whose base is *a*. The height of this parallelogram is $b \cdot \cos\gamma$. The area is base times height, and it equals abcosγ.

5) Cut the two figures and verify the equivalence of their areas. It means that:
$a^2 + b^2$ + 3 violet triangles (Area of left figure) = $c^2 + 2ab\cos\gamma$ + 3 violet triangles (Area of right figure)

By subtracting to both parts the 3 violet triangles, and moving $2ab\cos\gamma$ from right to left, one obtains the Cosine rule.