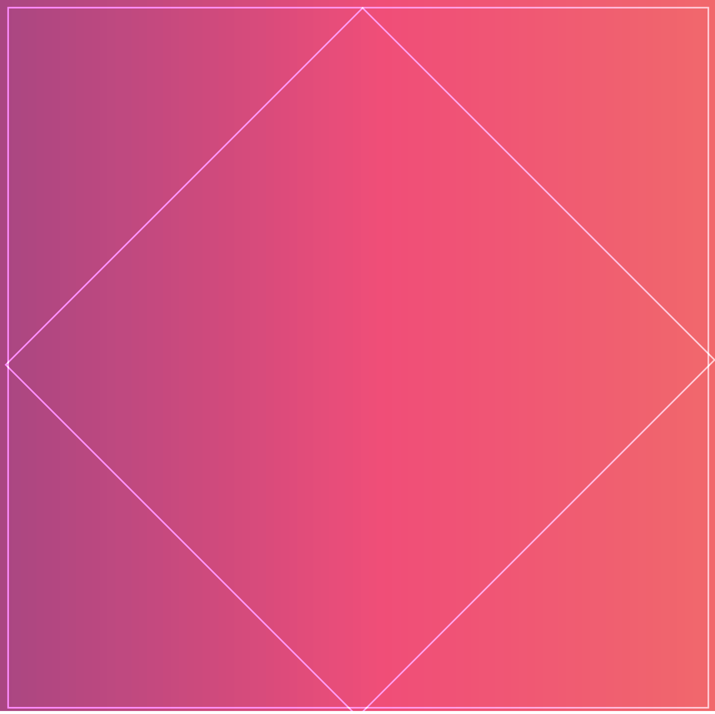
08



The Arithmetic Mean - Geometric Mean Inequality

Topic: Arithmetic
Theme: Illustration of the inequality between the arithmetic and geometric means.
Abilities: To make geometrical figures by following instructions.
Material: millimeter paper, scissors, a ruler
Level: Age 14/15

Math Reality

The Arithmetic Mean - Geometric Mean Inequality

For a geometrical interpretation of the AM – GM inequality, consider a rectangle with sides of length x and y, hence its perimeter is $2x + 2y$ and its area is xy. Similarly, the perimeter of a square of side $\sqrt{xy}$ is $4\sqrt{xy}$ and the same area as the rectangle. The simplest non-trivial case of the AM – GM inequality implies for the perimeters that $2x + 2y \geq 4\sqrt{xy}$ and that only the square has the smallest perimeter amongst all rectangles of equal area.

The inequality of arithmetic and geometric means states that the arithmetic mean of a list of non-negative real numbers is greater than or equal to the geometric mean of the same list.

An important practical application in financial mathematics is to computing the rate of return: the annualized return, computed via the geometric mean, is less than the average annual return, computed with the arithmetic mean (or equal if all returns are equal). This is important in analyzing investments, as the average return overstates the cumulative effect.

The arithmetic mean of a list of n numbers $x_1, x_2, \dots, x_n$ is the sum

$$\frac{x_1 + x_2 + \dots + x_n}{n}$$

The geometric mean is defined as the nth root of the product of n non-negative numbers. For a set of n non-negative numbers $x_1, x_2, \dots, x_n$, the geometric mean is defined as:

$$\sqrt[n]{x_1 \times x_2 \times \dots \times x_n}$$

For a list of n non-negative numbers $x_1, x_2, \dots, x_n$, AM – GM inequality is written as

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \times x_2 \times \dots \times x_n}$$

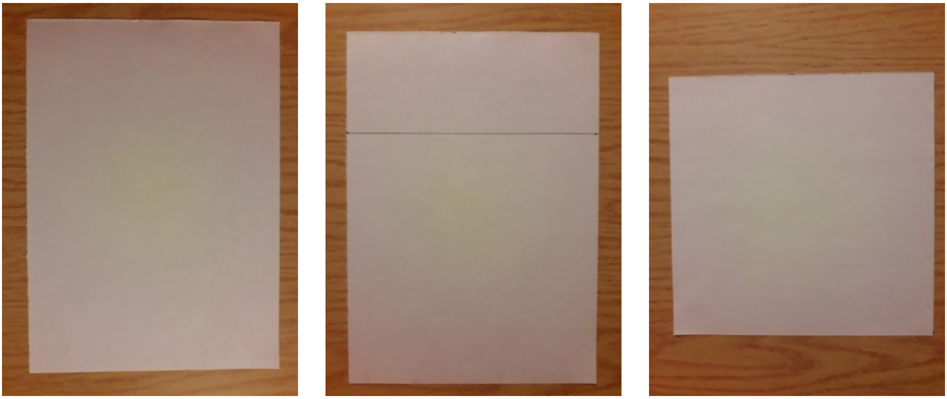
and equality holds if and only if $x_1 = x_2 = \dots = x_n$.

The case for two non-negative numbers a and b, is the statement that

$$\frac{a+b}{2} \geq \sqrt{ab}$$

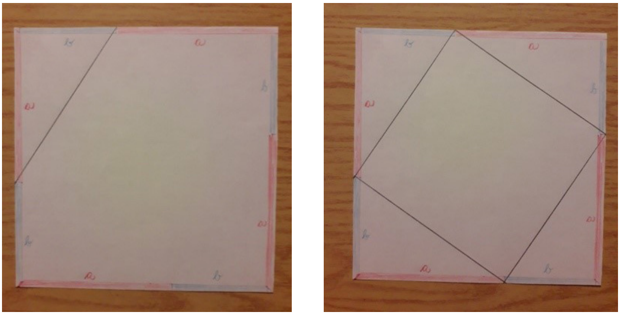
with equality if and only if $a = b$.

AM – GM inequality is a basic inequality, used to demonstrate other inequalities. Below you have a visual demonstration.

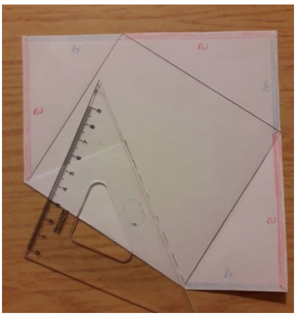


1- Make a square out of a paper sheet

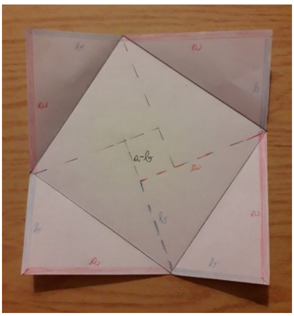
2- Divide every side into two segments with the length a and b.



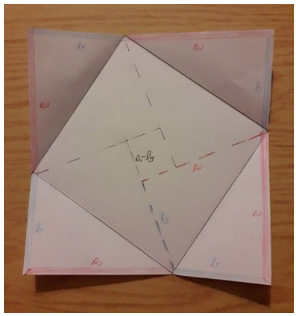
3- Draw a line from one point to the other, as seen in the pictures



4 - Fold the piece of paper along the obtained segments



5- Draw a dashed line along the longer side (of length a in our illustration).



6 - The area of the square with side $a + b$, which is $(a + b)^2$, is greater than the area of the 8 right triangles with catheti a and b, which is $8 \times ab/2$. We get the equality if and only if $a = b$.