LESSON SCENARIO 08: INEQUALITY OF MEANS

Topic: Arithmetic

Level: Age 14 -15

Foreknowledge: operations with fractions, operations with radicals, arithmetic mean, geometric mean, initial capital, final capital, simple interest and compound interest

Correlation: Financial mathematics, Art, Architecture

LEARNING OUTCOMES

- Calculation of arithmetic and geometric means in practical, concrete situations

TEACHING METHODS

- Practical work
- Hands-on activity
- Work in pairs

KEY WORDS

- arithmetic mean
- geometric mean
- inequality
- initial capital
- final capital
- simple interest
- compound interest

RESOURCES

- blackboard
- geometric instruments
- worksheets
- scissors
- projector
- laptop / computer
- pocket calculator

ACTIVITIES

ACTIVITY 1(5minutes)

The teacher presents the topic of the lesson and reminds the students the following concepts:

The word "average/mean" is found almost daily in people's discussions, in expressions such as: "average duration of people's life", "average life of a device", "average weight of a product". The average is a typical or central value of a lot of data. In order for the average size to have an objective character, it is necessary to choose the right type of mean (mathematical name for average). The most used means are: arithmetic mean; geometric mean; harmonic mean; square/quadratic mean.

The arithmetic mean of a list of *n* numbers $x_1, x_2, \dots, x_n$is the sum of the numbers divided by *n*:

$$\frac{x_1 + x_2 + \cdots + x_n}{n}.$$

The geometric mean is defined as the *n*th root of the product of *n*non-negative numbers. For a set of *n*non-negative numbers $x_1, x_2, \dots, x_n$, the geometric mean is defined as:

$$\sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}.$$

ACTIVITY 2 (15 minutes)

In order to see how important, it is to choose the right type of mean, the teacher presents the following practical activity and reminds the necessary notions.

The students are divided into three pairs: white, red and black.

The teacher reminds the students how to calculate the final capital in case of simple interest and compound interest:

- the final capital in case of simple interest $C_1 = C_0 \cdot \left(1 + \frac{p}{100}\right)$
- the final capital in case of compound interest $C_n = C_0 \cdot \left(1 + \frac{p}{100}\right)^n$

The teacher presents the problem to be solved:

A person deposits at the bank an amount (initial capital) of 1,000,000 lei over a period of 5 years, with annual interest rates varying as follows: 1%, 2%, 4%, 5%, 10%. We need to calculate the average annual rate (if applicable) and the final capital at the end of the 5 years.

The white teams will use the longer, but safer path, calculating simple interest rates for each year. The red teams will calculate the average interest rate using the arithmetic mean, and then they will calculate the final capital using the compound interest.

The black teams will calculate the average interest rate using the geometric mean, and then calculate the final capital using the compound interest rate.

The white teams calculate the final capital corresponding to each year:

Year 1: $C_1 = C_0 \cdot \left(1+\frac{p_1}{100}\right)$ = 1000000·1,01=1010000 lei

Year 2: $C_2 = C_1 \cdot \left(1+\frac{p_2}{100}\right)$= 1010000·1,02=1030200 lei

Year 3: $C_3 = C_2 \cdot \left(1+\frac{p_3}{100}\right)$= 1030200·1,04=1071408 lei

Year 4: $C_4 = C_3 \cdot \left(1+\frac{p_4}{100}\right)$= 1071408·1,05=1124978,40 lei

Year 5: $C_5 = C_4 \cdot \left(1+\frac{p_5}{100}\right)$= 1124978,40·1,10=1237476,24 lei.

The red teams calculate:

The average coefficient $1+\frac{p}{100}=\frac{1{,}01+1{,}02+1{,}04+1{,}05+1{,}10}{5}=1{,}044.$

The final capital: $C_5 = C_0 \cdot \left(1+\frac{p}{100}\right)^5$ = 1000000·$(1{,}044)^5$ = 1240230,745396224 lei.

The black teams calculate:

The average coefficient:$1+\frac{p}{100}=\sqrt[5]{1{,}01\cdot 1{,}02\cdot 1{,}04\cdot 1{,}05\cdot 1{,}10}=$ $\sqrt[5]{1{,}23747624}\approx 1{,}04353585.$

The final capital: $C_5 = C_0 \cdot \left(1+\frac{p}{100}\right)^5 \approx$ 1000000·$(1{,}04353585)^5 \approx$ 1000000·1,23747624 $\approx$ 1237476,24lei.

After each team presents its result, the conclusions are drawn. White and black teams achieved the same result, the correct one. The red team obtained an extra 2700 lei. Why? Because they used an additive operation (arithmetic mean) in the case of a multiplicative process (the final capital in the case of compound interest).

ACTIVITY 3 (15 minutes)

The teacher presents the inequality of means.

For a list of *n* non-negative numbers $x_1, x_2, \dots, x_n$, using mathematical notations, AM–GM, the inequality is written as:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n},$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

The case for two non-negative numbers a and b, is the statement that

$$\frac{a + b}{2} \geq \sqrt{ab}.$$

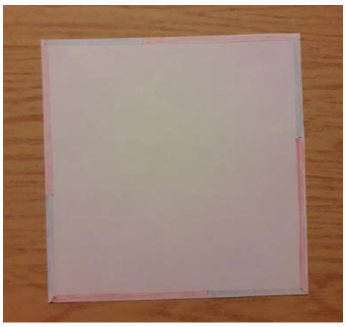
with equality if and only if $a = b$.

AM–GM inequality is a basic inequality, used to demonstrate other inequalities.

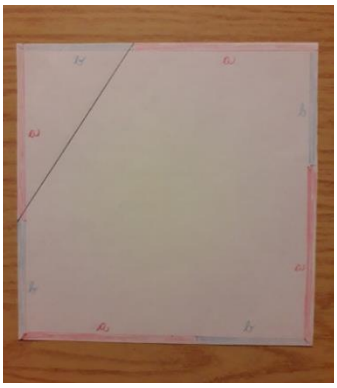
Below you have a visual demonstration:

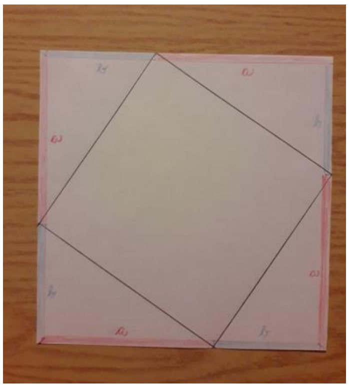
1. Make a square out of a paper sheet

2. Divide every side into two segments with the length *a* and *b*.

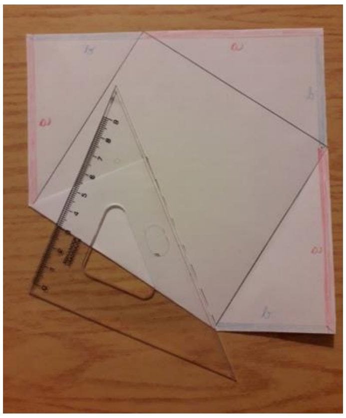


3. Draw a line from one point to the other, as seen in the pictures:

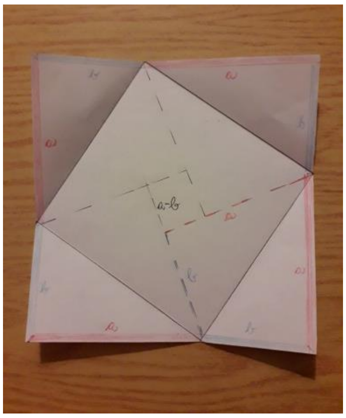




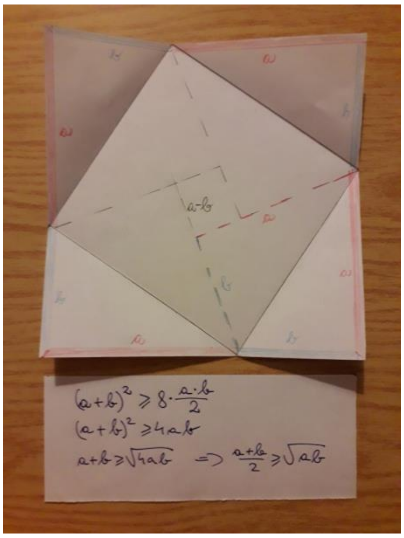
4. Fold the piece of paper alongside the segments resulted



5. Draw a dashed line alongside the longer side (of length a in our illustration).

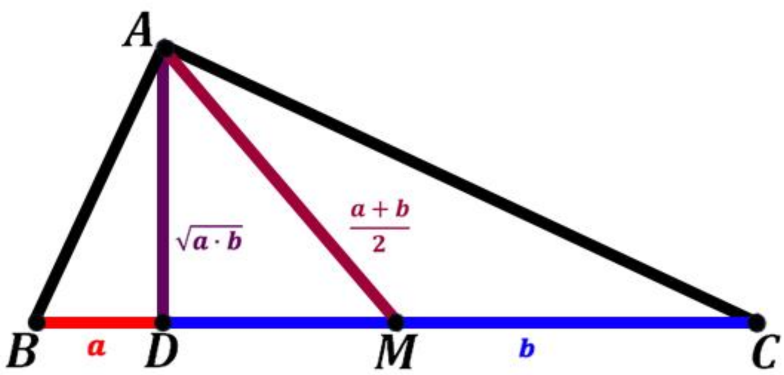


6. the mathematical area of the square with side $a+b$, $(a+b)^2$is 8 times greater than the one of a right triangles with catheters a and b, which is $8 \cdot \frac{a \cdot b}{2}$. And we get the inequality, only if $a=b$.



ACTIVITY 4 (5 minutes) (only performed if at least 15 minutes remain for evaluation)

The teacher presents a geometric interpretation of the inequality of means:



In any rectangular triangle the height corresponding to the hypotenuse is the geometric mean of the projections of the catheters on the hypotenuse, and the median corresponding to the hypotenuse is the arithmetic mean of the projections of the catheters on the hypotenuse. The length of the height is less than or equal to the length of the median.

EVALUATION

Evaluation sheet

Completesolutions are required for alltheproblems. The use ofthepocketcomputerisallowed.

(10p) 1. Determinethevalueoftruthofthefollowing sentence:

„$\frac{a+b+c}{3} > \sqrt[3]{abc}, \forall a, b, c \in (0, \infty)$" (true or false) and explain the choice you made.

(20p) 2. Calculatethearithmeticmeanofthenumbers 3, 4, 27, 64. Calculatethegeometricmeanofthenumbers: 3, 4, 27, 64. Comparetheresults.

(15p) 3. Calculate the artihmetical mean of the numbers: $3+\sqrt{8}$ and $3-\sqrt{8}$.

(15p) 4. Calculate the geometric mean of the numbers $3+\sqrt{8}$ and $3-\sqrt{8}$.

(30p) 5. A persondeposits at thebankanamount (initialcapital) of 1000000 leiover a period of 3 years, withannualinterestratesvarying as follows: 1%, 4%, 5%. Calculatethefinalcapital at theendofthe 3 years.

10 points are given ex officio.

Theworking time is 15 minutes.

Test solutions:

1.The statementisfalse. If *a* = *b* = *c*, then the inequalitybecomesequality.

2. $\frac{3+4+27+64}{4} = \frac{98}{4} = 24,5.$ $\sqrt[4]{3 \cdot 4 \cdot 27 \cdot 64} = \sqrt[4]{2^8 \cdot 3^4} = 2^2 \cdot 3 = 12$. AM>GM.

3. $\frac{(3+\sqrt{8})+(3-\sqrt{8})}{2} = \frac{6}{2} = 3.$

4. $\sqrt{(3+\sqrt{8})(3-\sqrt{8})} = \sqrt{9-8} = 1.$

5. **Solution 1**

Wecalculatethefinalcapitalcorresponding to eachyear:

Year 1: $C_1 = C_0 \cdot \left(1+\frac{p_1}{100}\right)$ = 1000000·1,01=1010000 lei

Year 2: $C_2 = C_1 \cdot \left(1+\frac{p_2}{100}\right)$ = 1010000·1,04=1050400 lei

Year 3: $C_3 = C_2 \cdot \left(1+\frac{p_3}{100}\right)$ = 1050400·1,05= 1102920 lei.

Solution 2

We calculate the average coefficient: $1+\frac{p}{100} = \sqrt[3]{1,01 \cdot 1,04 \cdot 1,05} = \sqrt[3]{1,10292} \approx 1,0331927199.$

We calculate the final capital: $C_3 = C_0 \cdot \left(1+\frac{p}{100}\right)^3 \approx 1000000 \cdot (1,0331927199)^3 \approx 1000000 \cdot 1,10292 \approx 1102920$ lei.

INCLUSIVENESS GUIDELINES

Every student is different and their needs for the material might vary. Below you will find several tips that could make mathematics lesson more inclusive for students who struggle with learning disorders.

- When giving assignments to classroom try to break them into small pieces of information. Avoid the double tasks in the instructions. Remember that in case of operations/exercises with multiple steps, it is critical to help learners decompose the steps.
- You can use checklists for your students to make sure they have done all the steps
- Make sure the font, line spacing, and alignment of your document is accessible for students with learning disorders. It is recommended to use a plain, evenly spaced sans serif font such as Arial and Comic Sans. Others: Verdana, Tahoma, Century Gothic and Trebuchet. Spacing should be 1.5 and try to avoid justification in the text.
- At the end of each activity, take some time to ask the students what they have learnt to acknowledge every step in their learning process
- Make sure that the material the students manipulate is easy enough to grasp
- While using different media (paper, computer and visual aids) choose different background than white which can be too bright for students with learning disorders. The best choice would be cream or soft pastel but try to test different colours to learn more about student's preference.
- To stimulate short and long-term memory prepare for all the students in the classroom an outline describing what they are going to learn on this lesson and finish it with a resume of what has been taught. In this way they will strengthen the ability to remember information.

EXAMPLE:

1. Start every lesson with a short "CHECK-IN"

- Today, we will study the topic (name of the topic)
- I will tell you about: (name 3 keywords connected with the topic)
- Then I will present exercises: (name the exercises from the student book)
- Then we will do exercises (explain the way student will be working: ex. together with teacher / in pairs /individually)
- Once the exercises will be done [To continue]

2. Then finish lesson with a short “CHECK-OUT”

- **During the lesson we learn about (topic of the lesson)**
- **The most important things were: (name 3 keywords connected with the topic)**
- **We were able to do… (tell about the work student done during the lesson)**
- **We will explore the topic next time when we will learn about (name the following topic)**

It is a small adjustment that will take 5 min from the lesson but can make a great difference in the way that the material will be remembered. Try to create this as a routine habit.