LESSON SCENARIO 06:
MONTY HALL PARADOX

Topic: conditional probability

Level: Age 15 - 18

Foreknowledge: events

Correlation: statistics, finance, gambling, artificial intelligence, machine learning, computer science, game theory

Time: 60 minutes

LEARNING OUTCOMES

- Use probability theory
- Find out Conditional probability

TEACHING METHODS

- VR technology
- individual work and pair work

KEY WORDS

- probability theory
- event
- conditional probability
- Monty Hall paradox

RESOURCES

- VR headsets
- blackboard
- laptop/ computer, pocket calculator, projector

ACTIVITIES

INTRODUCTION: RULES OF CONDUCT WHEN USING VR IN THE CLASSROOM (5 min)

The teacher starts discussion with the students asking them about the use of VR and their expectations in using VR in classroom.

After the discussion the teacher defines the work methods and rules of conduct for students regarding safety precautions for using VR headsets in the classroom and learning in virtual environment:

- listen to the teacher carefully
- remove physical obstacles before using VR
- always work in pair - never alone
- keep the device clean.

ACTIVITY 1 (5-10 min) INTRODUCTION TO THE LESSON

Form of work: frontal
Required accessories: blackboard or prepared PowerPoint
The teacher announces the subject of the lesson: Conditional Probabilities
The concept that will be used are reviewed.
Foreknowledge: random experiment, sample, event, elemental event, safe event, impossible event, incompatible events, opposite event, probability of an event, equiprobable events.
Theorem. If $\mathbf{E}$ – the multitude of all possible results of a random experiment is finite and all elementary events are equiprobable, then the probability of any event $\mathbf{A}$, relative to the considered experiment, is

$$\mathbf{P(A) = \frac{number\ of\ favourable\ cases}{number\ of\ possible\ cases}}.$$

Theorem. It is considered a random experiment with $\mathbf{E}$ – the multitude of possible results, finite and non-empty, and $\mathbf{A, B}$ – events related to the experiment considered. Then:

i)$\mathbf{P(A \cup B) = P(A) + P(B)}$, if $\mathbf{A, B}$ are incompatible;
ii)$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$, if $\mathbf{A, B}$ are not incompatible;
iii)$\mathbf{P(\overline{A}) = 1 - P(A)}$, where $\mathbf{\overline{A}}$ is the opposite event to $\mathbf{A}$.

ACTIVITY 2 (20 min) CONDITIONAL PROBABILITY

Form of work: frontal

Required accessories: blackboard or prepared PowerPoint

The teacher **announces the purpose of the new lesson, which is to assess the probability of an event that depends (or depends only in appearance) on another event already consumed.**

Problem 1.

In an urn there are 5 white balls and 9 black balls. Two balls are extracted successively, without turning the ball into the urn. Events shall be considered:

A: „the first ball extracted is black";

B: „the first ball extracted is white";

C: „the second ball extracted is white".

a) What is the probability of the event A?

b) What is the probability of the event C knowing that prior to its completion, event A was completed?

c) What is the probability of the event C knowing that the event was completed before it was realized B?

Solution.

a) $\mathbf{P(A)} = \frac{\text{number of favourable cases}}{\text{number of possible cases}}$

the number of possible cases is $5 + 9 = 14$

number of cases favorable to the achievement of the event A is 9, so $\mathrm{P(A)} = \frac{9}{14}$.

b) Following the completion of the event A, 5 white balls and 8 black balls remained in the urn, thus $\mathrm{P(C)} = \frac{5}{13}$.

c) Following the completion of the event B, 4 white balls and 9 black balls remained in the urn, thus $\mathrm{P(C)} = \frac{4}{13}$.

Definition. Be two events $\mathrm{A}, \mathrm{B} \in \mathcal{P}(\mathrm{E})$ so $\mathrm{P(A)} > 0$.

It defines the conditional probability of B given A by $\mathrm{P(B|A)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}}$.

This is $\mathrm{P(A \cap B)} = \mathrm{P(A)} \cdot \mathrm{P(B|A)}$ **achieved - the probability that both events A and B occur is equal to the probability of A multiplied by the conditional probability of B given A.**

Problem 2.

In an urn there are 5 white balls and 9 black balls. Extract two balls successively, without turning the ball into the urn. What is the probability of extracting a black ball followed by a white ball?

Solution.

Events shall be considered:

A: „the first ball extracted is black";

B: „the second ball extracted is white".

We get $\mathrm{P(A)} = \frac{9}{14}$ and $\mathrm{P(B|A)} = \frac{5}{13}$

Thus $\mathrm{P(A \cap B) = P(A) \cdot P(B|A)} = \frac{9}{14} \cdot \frac{5}{13} = \frac{45}{182}$.

Problem 3.

Two dice of different colors are thrown simultaneously. Events considered:

A: „the number on the first dice is less than that obtained on the second dice";

B: „the sum of the points earned on the two dice is less than or equal to 5".

What is the conditional probability of B given A?

Solution.

$\mathrm{P(B|A)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}}$.

Because the dice have different colors, the number of possible cases is $6 \cdot 6 = 36$.

$$\mathbf{A} = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$$

$$\mathbf{B} = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,2),(4,1)\}$$

$$\mathbf{A \cap B} = \{(1,2),(1,3),(1,4),(2,3)\}$$

Thus $\mathrm{P(A)} = \frac{15}{36}$, $\mathrm{P(B)} = \frac{10}{36}$, $\mathrm{P(A \cap B)} = \frac{4}{36}$ și $\mathrm{P(B|A)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}} = \frac{\frac{4}{36}}{\frac{15}{36}} = \frac{4}{15}$.

ACTIVITY 3 (15 min) CONDITIONAL PROBABILITY IN VR APPLICATION

The teacher assigns the task to the students.

Student:

- finds and selects the MONTY HALL PARADOX exercise on the exercise shelf
- solves tasks in VR application

Form of work: work in pairs

Required accessories: VR headset

COURSE OF ACTIVITY:

The teacher divides the students into pairs.

The teacher presents the Monty Hall problem to the students.

What is the Monty Hall Problem?

Also known as the as the *Monty Hall paradox*, the *three doors problem*, the *quizmaster problem*, and the problem of the *car and the goats*, the problem was introduced by bio-statistician Steve Selvin (1975) in a letter to the journal *The American Statistician*. Depending on what assumptions are made, it can be seen as mathematically identical to the Three Prisoners Problem of Martin Gardner (1959). It was named by Selvin after the stage-name of the actual quizmaster, Monty Halperin of the long-running 1960's TV show "Let's make a Deal". It's a famous paradox that has a solution that is so absurd, most people refuse to believe it's true.

The problem became world famous in 1990 with its presentation in the popular weekly column "Ask Marilyn" in *Parade* magazine. The author Marilyn vos Savant, was, according to the *Guiness Book of Records* at the time, the person with the highest IQ in the world. Rewriting in her own words a problem posed to her by a correspondent, Craig Whitaker, vos Savant asked the following:

"Suppose you're on a game show, and you're given the choice of three doors: behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?"

Student A carefully puts on his VR headset and opens the MONTY HALL PARADOX exercise in a virtual library in a VR application.

Pupil B (player) chooses a door, Pupil A (host) opens one of the remaining doors. Together they determine which door is with the correct probability for the second choice. The correct probabilities are $\frac{1}{3}$ for the door originally chosen, and $\frac{2}{3}$ for the door that should be chosen a second time.

Simple solutions

The solution presented by vos Savant in *Parade* shows the three possible arrangements of one car and two goats behind three doors and the result of staying or switching after initially picking door 1 in each case:

Behind door 1	Behind door 2	Behind door 3	Result if staying at door #1	Result if switching to the door offered
Goat	Goat	Car	Wins goat	Wins car
Goat	Car	Goat	Wins goat	Wins car
Car	Goat	Goat	Wins car	Wins goat

A player who stays with the initial choice wins in only one out of three of these equally likely possibilities, while a player who switches wins in two out of three.

Another way to understand the solution is to consider the two original unchosen doors together. The 2/3 chance of finding the car has not been changed by the opening of one of these doors because Monty, knowing the location of the car, is certain to reveal a goat. So, the player's choice after the host opens a door is no different than if the host offered the player the option to switch from the original chosen door to the set of both remaining doors. The switch in this case clearly gives the player a 2/3 probability of choosing the car.

Conditional probability by direct calculation

By definition, the conditional probability of winning by switching given the contestant initially picks door 1 and the host opens door 3 is the probability for the event "car is behind door 2 and host opens door 3" divided by the probability for "host opens door 3". These probabilities can be determined referring to the conditional probability table below. The conditional probability of winning by switching is $\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}} = \frac{2}{3}$.

<table>
<tr><td>Car hidden behind Door 3</td><td colspan="2">Car hidden behind Door 1</td><td>Car hidden behind Door 2</td></tr>
<tr><td colspan="4">Player initially picks Door 1</td></tr>
<tr><td>Host must open Door 2</td><td>Host randomly opens Door 2</td><td>Host randomly opens Door 3</td><td>Host must open Door 3</td></tr>
<tr><td>Probability 1/3</td><td>Probability 1/6</td><td>Probability 1/6</td><td>Probability 1/3</td></tr>
<tr><td>Switching wins</td><td>Switching loses</td><td>Switching loses</td><td>Switching wins</td></tr>
<tr><td colspan="2">On those occasions when the host opens Door 2, switching wins twice as often as staying</td><td colspan="2">On those occasions when the host opens Door 3, switching wins twice as often as staying</td></tr>
</table>

EVALUATION

1. I like the way of work in this lesson	1	2	3	4	5
2. This lesson was interesting	1	2	3	4	5
3. It is clear what I was supposed to learn in this lesson	1	2	3	4	5
4. The subject matter was clearly explained	1	2	3	4	5
5. I have learned the subject matter	1	2	3	4	5
6. I think I actively participated in this lesson	1	2	3	4	5
7. I was more active in this lesson than usually	1	2	3	4	5
8. By being active I contributed to the quality of the lesson	1	2	3	4	5
9. I was motivated for work in this lesson	1	2	3	4	5
10. I prefer using VR in lessons	1	2	3	4	5
111. Name two things you liked in this lesson:					
12. Name two things you didn`t like in this lesson:					

INCLUNSIVEESS GUIDELINES

Every student is different and their needs for the material might vary. Below you will find several tips that could make mathematics lesson more inclusive for students who struggle with learning disorders.

- When giving assignments to classroom try to break them into small pieces of information. Avoid the double tasks in the instructions. Remember that in case of operations/exercises with multiple steps, it is critical to help learners decompose the steps.
- You can use checklists for your students to make sure they have done all the steps
- Make sure the font, line spacing, and alignment of your document is accessible for students with learning disorders. It is recommended to use a plain, evenly spaced sans serif font such as Arial and Comic Sans. Others: Verdana, Tahoma, Century Gothic and Trebuchet. Spacing should be 1.5 and try to avoid justification in the text.
- At the end of each activity, take some time to ask the students what they have learnt to acknowledge every step in their learning process
- Make sure that the material the students manipulate is easy enough to grasp
- While using different media (paper, computer and visual aids) choose different background than white which can be too bright for students with learning disorders. The best choice would be cream or soft pastel but try to test different colors to learn more about student's preference.
- To stimulate short and long-term memory prepare for all the students in the classroom an outline describing what they are going to learn on this lesson and finish it with a resume of what has been taught. In this way they will strengthen the ability to remember information.

EXAMPLE:

1. Start every lesson with a short "CHECK-IN"
   - Today, we will study the topic (name of the topic)
   - I will tell you about: (name 3 keywords connected with the topic)
   - Then I will present exercises: (name the exercises form the student book)
   - Then we will do exercises (explain the way student will be working: ex. together with teacher / in pairs /individually)
   - Once the exercises will be done [To continue]

2. Then finish lesson with a short "CHECK-OUT"
   - During the lesson we learn about (topic of the lesson)

- **The most important things were: (name 3 keywords connected with the topic)**
- **We were able to do… (tell about the work student done during the lesson)**
- **We will explore the topic next time when we will learn about (name the following topic)**

It is a small adjustment that will take 5 min from the lesson but can make a great difference in the way that the material will be remembered. Try to create this as a routine habit.